QUALIFYING EXAM IN REAL VARIABLES
BAYLOR UNIVERSITY
FALL 2018

1. State *precisely* the following:

a) Fatou's Lemma

b) The Lebesgue Monotone Convergence Theorem

c) The Hahn-Banach Theorem

d) The Carathéodory Criterion

e) The Closed Graph Theorem

f) The Stone-Weierstrass Theorem

g) The Riesz Representation Theorem

h) The Ascoli-Arzelá Theorem

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $\{x \in \mathbb{R} : f(x) = \alpha\}$ is measurable for every $\alpha \in \mathbb{R}$. Must f be measurable? Justify your answer.

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Show that if U is a Borel set, then $f^{-1}(U)$ must be a Borel set.

4. Prove or find a counterexample: if E is a measurable set in $[0, 1]$ and χ_E has uncountable many discontinuities, then χ_E is not *Riemann* integrable. (Recall that $\chi_E(x) = 1$ if $x \in E$ and 0 otherwise.)

5. Is $C[0,1]$ the dual of any normed linear space? Justify your answer.

6. Let H be a Hilbert space and $\{u_n\}_{n=1}^{\infty}$ a countable orthonormal basis for H. Assume that $\{x_n\}$ is a sequence of vectors such that

$$\sum_{n=1}^{\infty} \|x_n - u_n\|^2 < 1.$$

Prove that the linear span of $\{x_n\}$ is dense in H.