QUALIFYING EXAM IN REAL VARIABLES
BAYLOR UNIVERSITY
SPRING 2018

1. State *precisely* the following:

a) The definition of a Borel set

b) The Lebesgue Dominated Convergence Theorem

c) The Hahn-Banach Theorem

d) The Baire Category Theorem

e) The Open Mapping Theorem

f) The Stone-Weierstrass Theorem

g) The Riesz Representation Theorem

h) The Ascoli-Arzelá Theorem

2. Prove or find a counterexample to the following:

a) If $A \subset [0, 1]$ is countable, then the Lebesgue measure of A is 0.

b) If $A \subset [0, 1]$ is uncountable, then the Lebesgue measure of A is greater than 0.

3. A real valued function f defined on an interval (a, b) is said to be *convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

whenever $x, y \in (a, b)$ and $t \in (0, 1)$.

a) Give an example of a non-constant, non-linear convex function.

b) Prove that if f is a non-constant convex function on $(a, b) \in \mathbb{R}$, then the set of local minima of f is a connected set where f is constant.

4.

(a) Let H be a Hilbert space, $K \subset H$ a closed subspace, and x a point in H. Show that there exists a unique y in K that minimizes the distance $\|x - y\|$ to x.

(b) Give an example to show that the conclusion can fail if H is an inner product space which is not complete.

5. Provide an example of a set $A \subset [0, 1]$ that contains no open intervals and such that the Lebesgue measure of A is *exactly* $\frac{1}{\pi}$.

6. Let $1 \leq p < q \leq \infty$.

(a) Prove $L^q[0, 1]$ is a subset of $L^p[0, 1]$.

(b) Provide an example of a function that is in $L^p[0, 1]$ that is not in $L^q[0, 1]$.