QUALIFYING EXAM IN REAL VARIABLES
BAYLOR UNIVERSITY
SPRING 2017

1. State *precisely* the following:

a) Fatou's Lemma

b) The Lebesgue Dominated Convergence Theorem

c) The Hahn-Banach Theorem

d) The Carathéodory Criterion

e) The Closed Graph Theorem

f) The Krein-Milman Theorem

g) The Riesz Representation Theorem

h) The Radon-Nikodym Theorem

2. Let X be a complete metric space with metric ρ. A map $f : X \to X$ is said to be *contracting* if for any two distinct points $x, y \in X$,

$$\rho(f(x), f(y)) < \rho(x, y) .$$

The map f is said to be *uniformly contracting* if there exists a constant $c < 1$ such that for any two distinct points $x, y \in X$,

$$\rho(f(x), f(y)) < c \cdot \rho(x, y) .$$

(a) Suppose that f is uniformly contracting. Prove that there exists a unique point $x \in X$ such that $f(x) = x$.

(b) Give an example of a contracting map $f : [0, \infty) \to [0, \infty)$ such that $f(x) \neq x$ for all $x \in [0, \infty)$.

3. What is the Cantor-Lebesgue function? How may it be used to construct a set S which is Lebesgue measurable but not a Borel set?

4. Let $\mathcal{C}_c^\infty(\mathbb{R})$ be the space of differentiable functions on $\mathbb{R}$ with compact support, and let $L^1(\mathbb{R})$ be the completion of $\mathcal{C}_c^\infty(\mathbb{R})$ with respect to the L^1 norm. Let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{h\to 0} \frac{1}{h} \int_{|y-x|<h} |f(y) - f(x)|\, dy = 0$$

for almost every x.

5. Let X be a Banach space.

(a) Define the *weak topology* on X by describing a basis for the topology.

(b) Let $A : X \to Y$ be a linear operator between Banach spaces that is continuous from the weak topology on X to the norm topology on Y. Show that the image $A(X) \subset Y$ is finite dimensional.

6. Let H be a Hilbert space and $\{u_n\}_{n=1}^\infty$ a countable orthonormal basis for H. Assume that $\{x_n\}$ is a sequence of vectors such that

$$\sum_{n=1}^{\infty} \|x_n - u_n\|^2 < 1.$$

Prove that the linear span of $\{x_n\}$ is dense in H.