Qualifying Exam

Dr. Herden Name: ______________________________

(A) State/Complete the indicated definitions:

(A1) The subgroup H of the group G is *normal*, if ... (**Give 3 different criteria!**)

(A2) Define the group $\langle S \mid K \rangle$. (What do the letters stand for?)

(A3) An integral domain R is a *UFD*, if ... (Write out all the relevant conditions!)

(A4) The following three properties of a nonzero R-module M are equivalent, and define a *semisimple* R*-module*:

(A5) The group G is *nilpotent*, if ...

(A6) An . . . polynomial is called *separable*, if . . .

An arbitrary polynomial is called *separable*, if . . .

(A7) The algebraic extension E/F is called *normal*, if . . .

(A8) A *Kummer extension* is . . .

(A9) For a finite degree field extension E/F and $x \in E$ define $m(x)$, $N(x)$ and $T(x)$.

(A10) In a category $\mathcal{C}$, a *coproduct* of objects M_j is . . . (Please include a diagram!)

(B) State/Complete the indicated results:

(B1) The Orbit-Counting Theorem.

(B2) The Jordan-Hölder Theorem.

(B3) An example of a prime ideal that is **not** maximal: (Include a short explanation!)

(B4) Fundamental Decomposition Theorem.

(B5) Artin-Wedderburn Theorem.

(B6) Cauchy's Theorem.

(B7) Fundamental Theorem of Galois Theory. (Parts (1), (2)(b) and (2)(d)! Include a diagram!)

(B8) The Steinitz Exchange.

(B9) Let B be the ring of algebraic integers of $\mathbb{Q}(\sqrt{d})$, d square-free.
Name an integral basis for B. (**There will be 2 cases!**)

(B10) Name 3 conditions on an R-module P that are equivalent to P being projective:

(C) Give a proof of the following statements:

(C1) Use the First Isomorphism Theorem for Groups to prove the Second Isomorphism Theorem.

(C2) Prove: Every proper ideal I of the ring R is contained in a maximal ideal.

(C3) Prove: If s is surjective and t and v are injective, then u is injective.

$$\begin{array}{ccccccc} D & \xrightarrow{e} & A & \xrightarrow{f} & M & \xrightarrow{g} & B \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ D' & \xrightarrow{e'} & A' & \xrightarrow{f'} & M' & \xrightarrow{g'} & B' \end{array}$$

(C4) Maschke's Theorem.

(C5) Prove: If E is a finite extension of F, then E is algebraic.

(C6) Let E/F be a finite field extension, such that F contains a primitive n^{th} root of unity ω with E a splitting field for $f(X) = X^n - a$ over F, $a \neq 0$.
Prove: E/F is a cyclic extension and the order of the Galois group G is a divisor of n.

(D) Work the following problems:

(D1) Show that $\langle a, b \mid a^4 = 1, b^2 = a^2, ab = ba^{-1} \rangle$ is a representation of the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$.

(D2) Identify the abelian group given by generators x_1, x_2, x_3 and relations

$$x_1 + 3x_2 + x_3 = 0,\ \ 3x_1 + x_2 + 5x_3 = 0,\ \ x_1 + 5x_2 + 7x_3 = 0.$$

(D3) Let $f(X) = X^4 - 2$ and $g(X) = X^4 - 32$.

- Find a splitting field K for $f(X)$ over $\mathbb{Q}$, and determine $[K : \mathbb{Q}]$.

- Show that K is also a splitting field for $g(X)$ over $\mathbb{Q}$.

- Show that $f(X)$ and $g(X)$ are both irreducible over $\mathbb{Q}$.

(D4)
- Show that $f(X) = X^3 - 3X + 1$ is irreducible over $\mathbb{Q}$.

- Find the Galois group G of f.

(D5) Consider the ring $R = \mathbb{Z}[\sqrt{-5}]$.

- Is R a UFD? (Give a short explanation! Two sentences will be sufficient!)

- Is R Dedekind? (Give a short explanation! Two sentences will be sufficient!) (**Hint:** Something from Section (B) might help!)

(D6) Let G be a group of order 208, and let H be a group of order 209.

- Show that G is not simple.

- Show that H is cyclic.

- How many elements of H are generators of H?