Qualifying Exam

Dr. Herden

Name: ________________________________

(A) State/Complete the indicated definitions:

(A1) The subgroup $P$ of the finite group $G$ is a *Sylow p-subgroup*, if . . .

(A2) Define the *outer semidirect product* of the groups $N$ and $H$.

(A3) A ring $R$ is a *Euclidean domain*, if . . .

(A4) A ring $R$ is *simple*, if . . .

(A5) Give three different definitions for the *Jacobson Radical* $J(R)$ of a ring $R$. 
(A6) For $F \subseteq E$ with $\alpha \in E$ give implicit and explicit definitions for the field $F(\alpha)$.

(A7) The algebraic extension $E/F$ is separable, if . . .

(A8) For a finite degree field extension $E/F$ and $x \in E$ define $m(x)$, $N(x)$ and $T(x)$.

(A9) The following two conditions on an $R$-module $M$ are equivalent, and define a Noetherian module:

(A10) In a category $C$, a coproduct of objects $M_j$ is . . . (Please include a diagram!)
(B) State/Complete the indicated results:

(B1) Orbit-Counting Theorem.

(B2) Jordan-Hölder Theorem for Groups.

(B3) An example of a prime ideal that is **not** maximal: (Include a short explanation!)

(B4) The Five Lemma. (Please include a diagram!)

(B5) Artin-Wedderburn Theorem.
(B6) Theorem of the Primitive Element.

(B7) Fundamental Theorem of Galois Theory. (Give two parts + a diagram!)

(B8) Galois’ Solvability Theorem.

(B9) Let $B$ be the ring of algebraic integers of $\mathbb{Q}(\sqrt{d})$, $d$ square-free. Name an integral basis for $B$. (There will be 2 cases!)

(B10) Name 2 conditions on an $R$-module $E$ that are equivalent to $E$ being injective:
(C) Give a proof of the following statements:

(C1) Prove: If $P$ is a nontrivial finite $p$-group, then $P$ has a nontrivial center.

(C2) Fundamental Decomposition Theorem.
(C3) Let $M$ be a nonzero $R$-module. Prove: If $M$ is a sum of simple modules, then $M$ is also a direct sum of simple modules. (You must not use that $M$ is semisimple!)

(C4) Transitivity of Algebraic Extensions.
(C5) Let $E/F$ be a finite field extension, such that $F$ contains a primitive $n^{th}$ root of unity $\omega$ with $E$ a splitting field for $f(X) = X^n - a$ over $F$, $a \neq 0$.
Prove: $E/F$ is a cyclic extension and the order of the Galois group $G$ is a divisor of $n$.

(C6) Prove: Every free module is projective.
(D) Work the following problems:

(D1) Show that \( \langle a, b \mid a^4 = 1, b^2 = a^2, ab = ba^{-1} \rangle \) is a representation of the quaternion group \( Q = \{ \pm 1, \pm i, \pm j, \pm k \} \).

(D2) Identify the abelian group given by generators \( x_1, x_2, x_3 \) and relations
\[
x_1 + 3x_2 + x_3 = 0, \ 3x_1 + x_2 + 5x_3 = 0, \ x_1 + 5x_2 + 7x_3 = 0.
\]
(D3) • Evaluate \([E : \mathbb{Q}]\) for \(E = \mathbb{Q}[\sqrt[3]{3}, \sqrt[9]{9}]\). (Explain your answer!)

• Evaluate \([E : \mathbb{Q}]\), where \(E\) is the splitting field for \(f(X) = X^{90} - 1\) over \(\mathbb{Q}\). (Explain your answer!)

(D4) • Show that \(f(X) = X^5 - 5X - 1\) is irreducible over \(\mathbb{Q}\). (Hint: Use a substitution \(X = Y + c\))

• Find the Galois group \(G\) of \(f\).
(D5) Consider the ring $R = \mathbb{Z}[\sqrt{-5}]$.

- Is $R$ a UFD? (Give a short explanation! Two sentences will be sufficient!)

- Is $R$ Dedekind? (Give a short explanation! Two sentences will be sufficient!)
  (Hint: Something from Section (B) might help!)

(D6) Let $G$ be a group of order 66. Show that $G$ is not simple.

- Let $G$ be a group of order 64. Can $G$ be simple? (Explain your answer!)
  (Hint: Something from Section (C) might help!)

- Let $G$ be a group of order 60. Can $G$ be simple? (Explain your answer!)