Qualifying Exam

Dr. Herden Name: ______________________________

(A) State/Complete the indicated definitions:

(A1) The subgroup P of the finite group G is a *Sylow p-subgroup*, if ...

(A2) Define the *outer* *semidirect product* of the groups N and H.

(A3) A ring R is a *Euclidean domain*, if ...

(A4) A ring R is *simple*, if ...

(A5) Give **three different definitions** for the *Jacobson Radical* $J(R)$ of a ring R.

(A6) For $F \leqslant E$ with $\alpha \in E$ give implicit and explicit definitions for the field $F(\alpha)$.

(A7) The algebraic extension E/F is *separable*, if ...

(A8) For a finite degree field extension E/F and $x \in E$ define $m(x)$, $N(x)$ and $T(x)$.

(A9) The following **two** conditions on an R-module M are equivalent, and define a *Noetherian module*:

(A10) In a category $\mathcal{C}$, a *coproduct* of objects M_j is ... (Please include a diagram!)

(B) State/Complete the indicated results:

(B1) Orbit-Counting Theorem.

(B2) Jordan-Hölder Theorem for Groups.

(B3) An example of a prime ideal that is **<u>not</u>** maximal: (Include a short explanation!)

(B4) The Five Lemma. (Please include a diagram!)

(B5) Artin-Wedderburn Theorem.

(B6) Theorem of the Primitive Element.

(B7) Fundamental Theorem of Galois Theory. (Give two parts + a diagram!)

(B8) Galois' Solvability Theorem.

(B9) Let B be the ring of algebraic integers of $\mathbb{Q}(\sqrt{d})$, d square-free. Name an integral basis for B. (There will be 2 cases!)

(B10) Name <u>2 conditions</u> on an R-module E that are equivalent to E being injective:

(C) Give a proof of the following statements:

(C1) Prove: If P is a nontrivial finite p-group, then P has a nontrivial center.

(C2) Fundamental Decomposition Theorem.

(C3) Let M be a nonzero R-module. Prove: If M is a sum of simple modules, then M is also a direct sum of simple modules. (You <u>**must not**</u> use that M is semisimple!)

(C4) Transitivity of Algebraic Extensions.

(C5) Let E/F be a finite field extension, such that F contains a primitive n^{th} root of unity ω with E a splitting field for $f(X) = X^n - a$ over F, $a \neq 0$.
Prove: E/F is a cyclic extension and the order of the Galois group G is a divisor of n.

(C6) Prove: Every free module is projective.

(D) Work the following problems:

(D1) Show that $\langle a, b \,|\, a^4 = 1, b^2 = a^2, ab = ba^{-1} \rangle$ is a representation of the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$.

(D2) Identify the abelian group given by generators x_1, x_2, x_3 and relations

$$x_1 + 3x_2 + x_3 = 0,\ 3x_1 + x_2 + 5x_3 = 0,\ x_1 + 5x_2 + 7x_3 = 0.$$

(D3) • Evaluate $[E:\mathbb{Q}]$ for $E=\mathbb{Q}[\sqrt[3]{5},\sqrt[7]{9}]$. (Explain your answer!)

• Evaluate $[E:\mathbb{Q}]$, where E is the splitting field for $f(X)=X^{90}-1$ over $\mathbb{Q}$. (Explain your answer!)

(D4) • Show that $f(X)=X^5-5X-1$ is irreducible over $\mathbb{Q}$. (**Hint:** Use a substitution $X=Y+c$!)

• Find the Galois group G of f.

(D5) Consider the ring $R = \mathbb{Z}[\sqrt{-5}]$.

- Is R a UFD? (Give a short explanation! Two sentences will be sufficient!)

- Is R Dedekind? (Give a short explanation! Two sentences will be sufficient!)
  (**Hint:** Something from Section (B) might help!)

(D6)
- Let G be a group of order 66. Show that G is not simple.

- Let G be a group of order 64. Can G be simple? (Explain your answer!)
  (**Hint:** Something from Section (C) might help!)

- Let G be a group of order 60. Can G be simple? (Explain your answer!)