Algebra Qualifying Examination
May 28, 2015

Dr. Dugas Name______________________________

(A) State the definition:

(A1) The ideal I of the ring R is maximal, if...

(A2) Let E be a field extension of the field F. The polynomial $f(x) \in F[x]$ splits over E, if...

(A3) The field extension E of the field F is normal, if...

(A4) Let F be a field of characteristic $p > 0$. The Frobenius map $\varphi : F \to F$ is...

(A5) The (finite or infinite) group G is a p-group, if...

(A6) The subgroup P of the finite group G is a p-Sylow subgroup of G, if...

(A7) The field extension $K \subseteq F$ is Galois, if...

(A8) The left R-module M is a free module, if...

(A9) The left R-module M is semisimple, if...

(A10) $R = \bigoplus_{i=1}^{n} R_i$ is a ring direct product, if...

(A11) The ring R is simple, if R is semisimple and ...

(A12) The left R-module is indecomposable, if ...

(A13) If R is a ring, then the Jacobson radical of R is $J(R) = ...$

(A14) The ideal I of the ring R is nilpotent, if...

(A15) The ring R is a UFD, if...

(A16) The R-module M is a projective R-module, if ...

(A17) The matrix N is the Smith Normal Form of the matrix M, if ...

(A18) φ is a contravariant functor between categories, if ...

(A19) The sequence $0 \to A \to B \to C \to 0$ is a short exact sequence, if ...

(A20) Define: T is the tensor product of the two modules A, B, if...

(B) State the indicated results:

(B1) Let J be a maximal ideal of the commutative ring R. Then R/J is a ....

(B2) Let E be a field extension of F and $g(x) \in F[x]$ irreducible over F. Let $\alpha, \beta \in E$ be roots of $g(x)$. Then there exists an F-homomorphism $\sigma : F[\alpha] \to F[\beta]$ such that.....

(B3) If every non-constant polynomial in $C[x]$ has a root in C, then C has no proper .... extensions.

(B4) Over a field of characteristic zero, every polynomial is....

(B5) The Orbit-Stabilizer Theorem.

(B6) The Sylow Theorems.

(B7) The Fundamental Theorem of Galois Theory.

(B8) The Primitive Element Theorem.

(B9) The First Isomorphism Theorem for left R-modules.

(B10) The Correspondence Theorem for left modules.

(B11) Schur's Lemma:

(B12) Jacobson's Theorem: If M is a semisimple R-module, $\varphi \in End_R(M)$ and $m_1, m_{2,} ...., m_k \in M$, then ...

(B13) If I, J are simple left ideals of R, then $I \cong J$ as R-modules or else ....

(B14) The Artin-Wedderburn's Theorem (on semisimple rings):

(B15) Maschke's Theorem:

(B16) The Jacobson radical $J(R)$ of the ring R is the largest ideal of R consisting entirely of .... elements.

(B17) R is Artinian and $J(R) = \{0\}$ if and only if ....

(B18) Baer's Injectivity Test Lemma.

(B19) The functor $Hom_R(\_, M)$ is exact, if ....

(B20) Every projective module is a direct summand of a ...

(B21) If D is a divisible abelian group and R some ring, then $Hom_Z(R, D)$ is a .... R-module.

(B22) Galois' Solvability Theorem.

(B23) $GF(p^m) \subseteq GF(p^n) \iff$ ...

(C) Prove the following results:

(C1) If M is a simple R-module, then $End_R(M)$ is a division ring.

(C2) Let G be a nilpotent group. Then G is solvable..

(C3) Let A be $S-R$-bimodule and B a left R-module. Then $A \otimes_R B$ is a left S-module.

(C4) If R is Artinian, then $J(R)$ is nilpotent.

(C5) "Transitivity" of algebraic field extensions.

(C6) Let $f(x) \in F[x]$ be a non-constant polynomial. Prove that there is a field extension E of F in which $f(x)$ has a root.

(C7) Let E be a field extension of the field F and let $A = \{a \in E : a \text{ is algebraic over } F\}$. Prove that A is a field.

(C8) The Orbit Stabilizer Theorem.

(C9) If G is a finite group and p a prime number dividing the order of G, then G has an element of order p.

(C10) The Second Isomorphism Theorem for Modules

(C11) Let I, J be simple left ideals of the rimg R such that $IJ \neq \{0\}$. Then $I \cong J$ as left R-modules.

(C12) A finitely generated module over a Noetherian ring is Noetherian.

(C13) If M is a projective R-module, then M is a direct summand of a free module.

(C14) Given modules A_R and ${}_RB$ and T_1 and T_2 tensor products of A and B, show that $T_1 \cong T_2$.

(C15) Prove: A direct summand of a flat R-module is flat.

Please work the following problems:

(D1) Show that $Q[\sqrt{3}, i] = Q[i + \sqrt{3}] =: E$. Find $[E : Q]$ and the minimal polynomial of $\alpha = i + \sqrt{3}$ over Q.

(D2) Let G be a group of order 33. Show that G is cyclic.

(D3) Let p be a prime and G be a finite group of order p^2. Show that G is abelian.

(D4) Find a Smith Normal Form for the integer matrix $A = \begin{bmatrix} 2 & 4 & 4 \\ -6 & 6 & 12 \\ 10 & -4 & -6 \end{bmatrix}$.

(D5) Let K be a field and V an infinite-dimensional vector space over K. Let $R = End_K(V)$ and $L = \{\varphi \in End_K(V) : \varphi(V)$ has finite dimension. Prove that L is an ideal of R. Explain why this proves: $End_K(V)$ is a simple ring if and only if $\dim_K(V)$ is finite.

(D6) Let G be a finite abelian group and $m = \text{lcm}\{|g| : 0 \neq g \in G\}$. Prove that there exists some element $a \in G$ with $|a| = m$. (Hint: Use the Fund. Thm. of ...)

(D7) Given a field K and $S = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} : a_{ij} \in K \right\}$.

Prove that $J(S) = \left\{ \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix} : a_{ij} \in K \right\}$. What is $S/J(S)$?

(D8) Let R be a ring. Show that $J(R/J(R)) = \{0\}$.

(D9) Prove: $Hom_Z(Q/Z, Q) = \{0\}$.

(D10) Prove: $Hom_Z(Z, Q) = \{q^* : q \in Q\}$ where $q^*(x) = xq$ for all $x \in Z$.

(D11) Combine (D9) and (D10) to show that $End_Z(Q) = \{q^* : q \in Q\}$ where $q^*(x) = xq$ for all $x \in Q$.

(D12) Let R be a commutative ring and $I, J \trianglelefteq R$. Show that $(R/I) \otimes_R (R/J) \cong R/(I + J)$.

(D13) Let $Z[x]$ be the ring of integer polynomials. Prove: $Q \otimes_Z Z[x] \cong Q[x]$.

(D14) Explain why $f(x) = 7x^{11} - 9x + 3$ is irreducible over Q.