

BAYLOR UNIVERSITY

Department of Mathematics

Ph.D Comprehensive Examination: Applied Math Part II Summer/Fall, 2015

INSTRUCTIONS:

- Do **3** of the following 6 problems. This part of the exam takes up to 120 minutes to complete.
 - Write your results clearly so that a scanned copy can be emailed.
 - You will be graded on how you arrived at the final answer. **Show your detailed work.**
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1. Consider the numerical solution of the IVP

$$y' = f(t, y), \quad t \geq t_0, \quad y(t_0) = y_0, \quad (1.1)$$

where f is sufficiently smooth. Denote $t_{n+1} = t_n + h$, $n = 0, 1, 2, \dots$, $0 < h \ll 1$, and assume that $y_m \approx y(t_m)$ for $m = 1, 2, \dots$. Let

$$y_{n+1} = y_n + \frac{h}{2} [f(t_n, y_n) + f(t_{n+1}, y_{n+1})], \quad n = 0, 1, 2, \dots \quad (1.2)$$

- (a) Show that (1.2) is a consistent approximation of (1.1).
(b) Show that the numerical method (1.2) is convergent. Derive its order of convergence.
2. Given the following order p and convergent multistep method for solving (1.1):

$$\sum_{m=0}^s \alpha_m y_{n+m} = h \sum_{m=0}^s \beta_m f(t_{n+m}, y_{n+m}), \quad n = 0, 1, 2, \dots, \quad \alpha_s = 1, \quad s \geq 1. \quad (2.1)$$

Derive a proper Milne device for assessing its local error.

3. Let $T = (t_{k-j})_{k,j=1}^n$ be an $n \times n$ TST matrix with $t_{-n+1} = \dots = t_{-2} = 0$, $t_{-1} = \beta$, $t_0 = \alpha$, $t_1 = \beta$, $t_2 = \dots = t_{n-1} = 0$.
- (a) Show that the eigenvalues of T are

$$\lambda_j = \alpha + 2\beta \cos\left(\frac{\pi j}{n+1}\right), \quad j = 1, 2, \dots, n.$$

- (b) Use the above result to show that there exists a conditionally stable explicit finite difference method for solving the IBVP:

$$\begin{aligned}u_t &= u_{xx}, & -1 < x < 1, & t > t_0; \\u(-1, t) &= u(1, t) = 0, & t &\geq t_0; \\u(x, 0) &= \phi(x), & -1 < x < 1,\end{aligned}$$

where ϕ is sufficiently smooth on $(-1, 1)$. If the stability is conditional, find the condition.

4. Consider the two-dimensional Poisson equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u(x, y) = f(x, y), \quad (x, y) \in \Omega, \quad (4.1)$$

where $\Omega = \{(x, y) \mid 0 < x, y < 1\}$, together with the homogeneous Dirichlet boundary condition

$$u(x, y)|_{\partial\Omega} = 0. \quad (4.2)$$

Let Ω_h be a uniform mesh region over Ω and its boundary $\partial\Omega$ with a uniform step size $0 < h \ll 1$, $(n+1)h = 1$, used in both directions. Consider the standard 5-point scheme for solving the problem:

$$u_{k-1,j} + u_{k+1,j} + u_{k,j-1} + u_{k,j+1} - 4u_{k,j} = h^2 f_{k,j}, \quad 1 \leq k, j \leq n. \quad (4.3)$$

- (a) Show that (4.3) can be rewritten as a matrix equation

$$Au = h^2 f.$$

- (b) Show that, following a standard order arrangement of the components in u , the matrix A has the eigenvalues

$$\lambda_{\alpha,\beta} = -4 \left[\sin^2 \left(\frac{\alpha\pi}{2(n+1)} \right) + \sin^2 \left(\frac{\beta\pi}{2(n+1)} \right) \right]$$

and eigenvectors $v_{\alpha,\beta} = \{v_{\alpha,\beta,i,j}\}$, in which

$$v_{\alpha,\beta,i,j} = \sin \left(\frac{i\alpha\pi}{n+1} \right) \sin \left(\frac{j\beta\pi}{n+1} \right), \quad i, j = 1, 2, \dots, n,$$

for $\alpha, \beta = 1, 2, \dots, n$.

5. Let $y = f(x)$, $x \in \mathbb{R}$, be sufficiently smooth and

$$\Delta_+ f(x) = \frac{f(x+p) - f(x)}{p}, \quad \Delta_- f(x) = \frac{f(x) - f(x-q)}{q},$$

where $0 < p, q \ll 1$, $p \neq q$.

- (a) Show that $\Delta_+\Delta_-f(x) \neq \Delta_-\Delta_+f(x)$.
- (b) Show that neither $\Delta_+\Delta_-f(x)$ nor $\Delta_-\Delta_+f(x)$ is a consistent approximation of the derivative $f'(x)$, $x \in \mathbb{R}$.
- (c) Derive a consistent finite difference formula approximating $f'(x)$ utilizing the above steps p , q for $x \in \mathbb{R}$. Show the order of convergence.

6. Consider the Cauchy problem

$$w_t = w_{xx}, \quad -\infty < x < \infty, \quad t > 0; \quad w(x, 0) = \phi(x), \quad -\infty < x < \infty. \quad (6.1)$$

Use the von Neumann analysis to discuss the numerical stability of the following finite difference scheme,

$$w_k^{n+1} = w_k^n + \mu (w_{k+1}^{n+1} - 2w_k^{n+1} + w_{k-1}^{n+1}), \quad n \geq 0; \quad u_k^0 = \phi_k,$$

where $\mu > 0$ is the Courant number and ϕ is sufficiently smooth, for solving (6.1).