BAYLOR UNIVERSITY
Department of Mathematics

Ph.D Comprehensive Examination: Applied Math Part II
Summer, 2014

INSTRUCTIONS:

- Do **3** of the following 6 problems. This part of the exam takes up to 120 minutes to complete.
- Write your results clearly so that a scanned copy can be emailed.
- You will be graded on how you arrived at the final answer. **Show your detailed work**.

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1. Consider the numerical solution of the IVP

$$y' = f(t, y), \quad t \geq t_0,\ y(t_0) = y_0. \tag{1.1}$$

   (a) Show that the following trapezoidal rule method is convergent.

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right], \quad n = 0, 1, 2, \ldots$$

   (b) Derive the order of convergence of the above method.

2. Given the following order p and convergent multistep method for solving (1.1):

$$\sum_{m=0}^{s} a_m y_{n+m} = h \sum_{m=0}^{s} b_m f(t_{n+m}, y_{n+m}), \quad n = 0, 1, 2, \ldots,\ a_s = 1. \tag{2.1}$$

   Derive a proper Milne device for assessing its local error.

3. Let $T = (t_{k-j})_{k,j=1}^{n}$ be an $n \times n$ TST matrix with $t_0 = \alpha > 0$, $t_{-1} = t_1 = \beta > 0$.

   (a) Show that the eigenvalues of T are

$$\lambda_j = \alpha + 2\beta \cos\left(\frac{\pi j}{n+1}\right), \quad j = 1, 2, \ldots, n.$$

   (b) Use the above result to show that there exists a stable explicit finite difference method for solving the IBVP:

$$\begin{aligned} &u_t = u_{xx}, \quad -1 < x < 1,\ t > t_0; \\ &u(-1, t) = u(1, t) = 0, \quad t \geq t_0; \\ &u(x, 0) = \phi(x), \quad -1 < x < 1, \end{aligned}$$

   where ϕ is sufficiently smooth on $(-1, 1)$. If the stability is conditional, find the condition.

4. Consider the solution of the linear system

$$Ax = b, \tag{4.1}$$

where $A \in \mathbb{R}^{n\times n},\ n \gg 1$.

(a) Derive the Jacobi, Gauss-Seidel and SOR iterative schemes, respectively, for solving (4.1).

(b) Under what condition the above-mentioned iterative methods converge?

(c) Show that if A is irreducible and strictly diagonally dominant then the Jacobi method converges.

5. Let $y = f(x),\ x \in \mathbb{R}$, be sufficiently smooth and

$$\Delta_+ f(x) = \frac{f(x+p) - f(x)}{p}, \quad \Delta_- f(x) = \frac{f(x) - f(x-q)}{q},$$

where $0 < p, q \ll 1,\ p \neq q$.

(a) Show that in general $\Delta_+\Delta_- f(x) \neq \Delta_-\Delta_+ f(x)$.

(b) Show that neither $\Delta_+\Delta_- f(x)$ nor $\Delta_-\Delta_+ f(x)$ is a consistent approximation of the derivative $f''(x),\ x \in \mathbb{R}$.

(c) Derive a consistent finite difference formula approximating $f''(x)$ utilizing the above steps $p,\ q$ for $x \in \mathbb{R}$. Show the order of convergence.

6. Consider the Cauchy problem

$$u_t = u_{xx}, \ -\infty < x < \infty,\ t > 0; \quad u(x,0) = \phi(x), \ -\infty < x < \infty. \tag{6.1}$$

Use the von Neumann analysis to discuss the numerical stability of the finite difference scheme,

$$u_k^{n+1} = u_k^n + \mu\left(u_{k+1}^{n+1} - 2u_k^{n+1} + u_{k-1}^{n+1}\right), \quad n \geq 0; \quad u_k^0 = \phi_k,$$

where $\mu > 0$ is the Courant number and ϕ is sufficiently smooth, for solving (6.1).