Ph.D. Qualifying Exam

Applied Mathematics I

Solve FOUR of the following problems.

1. (a) Show that $L(f)(x) = \int_0^x f(t)dt$ defines a continuous linear mapping from $C([0,1])$ into itself, and find its operator norm.

   (b) Is the mapping L defined above compact? Why or why not?

2. (a) Let V be an inner product space. State what it means for $\{x_i\}_{i=0}^{\infty}$ to be a *complete orthogonal sequence* in V.

   (b) Is $\{(1,1,0,0,0,\dots),(1,-1,0,0,0,\dots),(0,0,1,1,0,\dots),(0,0,1,-1,0\dots),\dots\}$ a complete orthogonal sequence in ℓ^2? Why or why not?

3. (a) State the Banach fixed point theorem.

   (b) Define the (nonlinear!) operator T mapping $C[0,1]$ into itself by

   $$(Tu)(t) = \int_0^t (u(x))^2\,dx,$$

   and show that, although T is not a contraction on the closed unit ball, it is on the closed ball of radius $\frac{1}{4}$.

   (c) Does there exist a continuous function u such that $Tu = u$?

4. Let V be an inner product space.

   (a) State what it means for a sequence to converge *weakly* in V.

   (b) Show that, if V is also finite-dimensional and Hilbert, that every weakly convergent sequence is strongly convergent.

   (c) Give an example of a weakly convergent sequence in an infinite-dimensional space that does not converge strongly. You should demonstrate the weak and lack of strong convergence.

5. (a) State the Riesz Representation Theorem for a Hilbert space H over $\mathbb{C}$.

   (b) Because this theorem associates to each member of H' a member of H, it defines a mapping T from H' into H. Show that the following properties hold for all $f, g \in H$ and all $\alpha \in \mathbb{C}$:

   $$T(f+g) = T(f) + T(g)$$
   $$T(\alpha f) = \overline{\alpha} T(f)$$
   $$\|T(f)\| = \|f\|$$