Solve FOUR of the following problems.

1. (a) Show that \( L(f)(x) = \int_0^x f(t)dt \) defines a continuous linear mapping from \( C([0, 1]) \) into itself, and find its operator norm.

(b) Is the mapping \( L \) defined above compact? Why or why not?

2. (a) Let \( V \) be an inner product space. State what it means for \( \{x_i\}_{i=0}^\infty \) to be a complete orthogonal sequence in \( V \).

(b) Is \( \{(1, 1, 0, 0, \ldots), (1, -1, 0, 0, \ldots), (0, 0, 1, 1, 0, \ldots), (0, 0, 1, -1, 0, \ldots), \ldots\} \) a complete orthogonal sequence in \( \ell^2 \)? Why or why not?

3. (a) State the Banach fixed point theorem.

(b) Define the (nonlinear!) operator \( T \) mapping \( C[0, 1] \) into itself by

\[
(Tu)(t) = \int_0^t (u(x))^2\,dx,
\]

and show that, although \( T \) is not a contraction on the closed unit ball, it is on the closed ball of radius \( \frac{1}{4} \).

(c) Does there exist a continuous function \( u \) such that \( Tu = u \)?

4. Let \( V \) be an inner product space.

(a) State what it means for a sequence to converge weakly in \( V \).

(b) Show that, if \( V \) is also finite-dimensional and Hilbert, that every weakly convergent sequence is strongly convergent.

(c) Give an example of a weakly convergent sequence in an infinite-dimensional space that does not converge strongly. You should demonstrate the weak and lack of strong convergence.

5. (a) State the Riesz Representation Theorem for a Hilbert space \( H \) over \( \mathbb{C} \).

(b) Because this theorem associates to each member of \( H' \) a member of \( H \), it defines a mapping \( T \) from \( H' \) into \( H \). Show that the following properties hold for all \( f, g \in H \) and all \( \alpha \in \mathbb{C} \):

\[
T(f + g) = T(f) + T(g)
\]
\[
T(\alpha f) = \overline{\alpha} T(f)
\]
\[
\|T(f)\| = \|f\|
\]