1. Let $y = f(x), \ x \in \mathbb{R}$, be sufficiently smooth and

$$
\Delta_+ f(x) = \frac{f(x + p) - f(x)}{p}, \ \Delta_- f(x) = \frac{f(x) - f(x - q)}{q},
$$

where $0 < p, q \ll 1$.

(a) Show that in general the commutator

$$
[\Delta_+, \Delta_-] \neq 0.
$$

(b) Derive the truncation errors of $\Delta_+ f(x)$ and $\Delta_- f(x)$ when they are used to approximate the derivative $f'(x), \ x \in \mathbb{R}$.

2. Let $y = f(x), \ x \in \mathbb{R}$, be sufficiently smooth and

$$
\Delta_+ f(x) = \frac{f(x + h) - f(x)}{h}, \ \Delta_- f(x) = \frac{f(x) - f(x - h)}{h},
$$

$$
\Delta_0 f(x) = \frac{f(x + h/2) - f(x - h/2)}{h}, \ x \in \mathbb{R},
$$

where $0 < h \ll 1$. Further, define

$$
D f(x) = f'(x), \ I f(x) = f(x), \ x \in \mathbb{R}.
$$

Show that

(a) $hD = \ln (I + h\Delta_+)$.

(b) $hD = -\ln (I - h\Delta_-)$. 
(c) \[ hD = \ln \left( \frac{h}{2} \Delta_0 + \sqrt{I + \frac{h^2}{4} \Delta_0^2} \right). \]

3. Let \( t > 0 \) and \( C, D \in \mathbb{R}^{n \times n} \) be diagonal.

   (a) How can you compute the matrix exponential \( E(t) = e^{tC} \)?
   (b) Do matrix exponentials \( e^{tC}, e^{tD} \) commute? Prove your result.

4. Let
   \[
   A = \begin{bmatrix}
   -2 & 1 & 0 \\
   1 & -2 & 1 \\
   0 & 1 & -2
   \end{bmatrix}.
   \]

   (a) Compute the spectral radius and spectral norm of the matrix \( A \).
   (b) Let \( Ax = b \) be a linear system. Derive explicitly the iterative matrix for solving it if a Gauss-Jacobi method is utilized.

5. Let \( 0 < h \ll 1 \). The forward Euler scheme on a uniform mesh,
   \[
y_{n+1} - y_n = g(t_n, y_n)h, \quad n = 0, 1, 2, \ldots; \quad y_0 = \alpha,
   \]
   is often used for solving the initial value problem
   \[
y'(t) = g(t, y(t)), \quad t > t_0; \quad y(t_0) = \alpha,
   \]
   where \( g \) is sufficiently smooth for \( t > t_0 \) and satisfies the Lipschitz condition for the second variable.

   (a) Derive the error equation between \( \epsilon_{n+1} \) and \( \epsilon_n \) for (4.1).
   (b) Conduct a convergence analysis for (4.1).

6. Given the boundary value problem
   \[
u_{xx} = f(x), \quad 0 < x < 1; \quad u_x(0) = \alpha, \quad u_x(1) = \beta,
   \]
   where \( \alpha, \beta \) are constants and \( f \) is sufficiently smooth. Let \( \Omega_h \) be a uniform mesh of \( N + 2 \) points superimposed on \([0, 1]\).

   (a) Design at least two different discretizations of the boundary conditions in (5.1). What can be your order of accuracy, respectively?
(b) Suppose that a standard central difference is used to approximate the differential equation. Can the linear systems obtained be singular? Prove your results.

7. Consider the Cauchy problem

\[ u_t = u_{xx}, \quad -\infty < x < \infty, \quad t > 0; \quad u(x, 0) = \phi(x), \quad -\infty < x < \infty. \]  \hfill (6.1)

Use the von Neumann analysis to discuss the numerical stability of the finite difference scheme,

\[ u_{k}^{n+1} = u_k^n + \mu \left( u_{k+1}^n - 2u_k^n + u_{k-1}^n \right), \quad n \geq 0; \quad u_k^0 = \phi_k, \]

where \( \mu > 0 \) is the Courant number and \( \phi \) is sufficiently smooth, for solving (6.1).