BAYLOR UNIVERSITY
Department of Mathematics

Ph.D Comprehensive Examination: Applied Math Part II
Summer, 2013

INSTRUCTIONS:

- Do **4** of the following 7 problems. This part of the exam takes up to 120 minutes to complete.
- Write your results clearly so that a scanned copy can be emailed.
- You will be graded on how you arrived at the final answer. **Show your detailed work**.

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1. Let $y = f(x),\ x \in \mathbb{R}$, be sufficiently smooth and

$$\Delta_+ f(x) = \frac{f(x+p) - f(x)}{p}, \quad \Delta_- f(x) = \frac{f(x) - f(x-q)}{q},$$

where $0 < p, q \ll 1$.

   (a) Show that in general the commutator

$$[\Delta_+, \Delta_-] \neq 0.$$

   (b) Derive the truncation errors of $\Delta_+ f(x)$ and $\Delta_- f(x)$ when they are used to approximate the derivative $f'(x),\ x \in \mathbb{R}$.

2. Let $y = f(x),\ x \in \mathbb{R}$, be sufficiently smooth and

$$\Delta_+ f(x) = \frac{f(x+h) - f(x)}{h}, \; \Delta_- f(x) = \frac{f(x) - f(x-h)}{h},$$
$$\Delta_0 f(x) = \frac{f(x+h/2) - f(x-h/2)}{h}, \quad x \in \mathbb{R},$$

where $0 < h \ll 1$. Further, define

$$\mathcal{D} f(x) = f'(x), \quad I f(x) = f(x), \quad x \in \mathbb{R}.$$

Show that

   (a) $h\mathcal{D} = \ln\left(I + h\Delta_+\right).$

   (b) $h\mathcal{D} = -\ln\left(I - h\Delta_-\right).$

(c) $h\mathcal{D} = \ln\left(\frac{h}{2}\Delta_0 + \sqrt{I + \frac{h^2}{4}\Delta_0^2}\right)$.

3. Let $t > 0$ and $C,\ D \in \mathbb{R}^{n\times n}$ be diagonal.

   (a) How can you compute the matrix exponential $E(t) = e^{tC}$?

   (b) Do matrix exponentials $e^{tC},\ e^{tD}$ commute? Prove your result.

4. Let

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

   (a) Compute the spectral radius and and spectral norm of the matrix A.

   (b) Let $Ax = b$ be a linear system. Derive explicitly the iterative matrix for solving it if a Gauss-Jacobi method is utilized.

5. Let $0 < h \ll 1$. The forward Euler scheme on a uniform mesh,

$$y_{n+1} - y_n = g(t_n, y_n)h,\ n = 0, 1, 2, \ldots;\quad y_0 = \alpha, \tag{4.1}$$

   is often used for solving the initial value problem

$$y'(t) = g(t, y(t)),\ t > t_0;\quad y(t_0) = \alpha,$$

   where g is sufficiently smooth for $t > t_0$ and satisfies the Lipschitz condition for the second variable.

   (a) Derive the error equation between ϵ_{n+1} and ϵ_n for (4.1).

   (b) Conduct a convergence analysis for (4.1).

6. Given the boundary value problem

$$u_{xx} = f(x),\ 0 < x < 1;\quad u_x(0) = \alpha,\ u_x(1) = \beta, \tag{5.1}$$

   where $\alpha,\ \beta$ are constants and f is sufficiently smooth. Let Ω_h be a uniform mesh of $N + 2$ points superimposed on $[0, 1]$.

   (a) Design at least two different discretizations of the boundary conditions in (5.1). What can be your order of accuracy, respectively?

(b) Suppose that a standard central difference is used to approximate the differential equation. Can the linear systems obtained be singular? Prove your results.

7. Consider the Cauchy problem

$$u_t = u_{xx}, \ -\infty < x < \infty, \ t > 0; \quad u(x,0) = \phi(x), \ -\infty < x < \infty. \tag{6.1}$$

Use the von Neumann analysis to discuss the numerical stability of the finite difference scheme,

$$u_k^{n+1} = u_k^n + \mu \left(u_{k+1}^n - 2u_k^n + u_{k-1}^n \right), \quad n \geq 0; \quad u_k^0 = \phi_k,$$

where $\mu > 0$ is the Courant number and ϕ is sufficiently smooth, for solving (6.1).