Topology **Ph.D. Qualifying Exam** **August 16, 2013**

(1) (3 points each) Complete each of the following definitions.

(a) A set is *well ordered* ...

(b) A *topology* ...

(c) A T_1*-space* ...

(d) The *quotient topology* on a set Y induced by a function f from a topological space X onto Y is ...

(e) Suppose $\{X_a : a \in A\}$ is a collection of topological spaces. The *product topology* on $\prod_A X_a$ is ...

(f) A topological space is *first countable* ...

(g) A path f in a space X is said to be *path homotopic* to a path f' in X if and only if ...

(h) Suppose x_0 is a point of a topological space X. The *fundamental group* of X relative to the *base point* x_0 is ...

(i) A map $f : X \to Y$ is said to be a *homotopy equivalence* if and only if ...

(j) An *exact sequence* ...

(2) (3 points each) Give an example of each of the following, or state that no such example exists. You need not show any work.

(a) a sequence, $H_1, H_2, H_3, ...$ of subsets of $\mathbb{R}$ such that $\overline{H}_1 \cup \overline{H}_2 \cup \overline{H}_3 \cup ... \neq \overline{H_1 \cup H_2 \cup H_3 \cup ...}$

(b) two Hausdorff spaces whose product fails to be Hausdorff.

(c) a countable basis for the usual topology on $\mathbb{R}$

(d) a connected set that fails to be locally connected.

(e) a closed and bounded subset of a metric space that fails to be compact.

(f) two non-homeomorphic spaces X and Y and points $x_0 \in X$ and $y_0 \in Y$ such that $\Pi_1(X, x_0)$ and $\Pi_1(Y, y_0)$ are isomorphic.

(g) two path-connected, homeomorphic spaces X and Y and points $x_0 \in X$ and $y_0 \in Y$ such that $\Pi_1(X, x_0)$ and $\Pi_1(Y, y_0)$ are not isomorphic.

(h) a space X and two points, x_0 and x_0' of X such that $\Pi_1(X, x_0)$ and $\Pi_1(X, x_0')$ are not isomorphic.

(i) a covering map for S^1 that is not a homeomorphism.

(j) A space X for which $H_1(X)$ is trivial, but $H_2(X)$ is nontrivial.

(3) (10 points each) Prove three of the following.

(a) Suppose $G = \{X_a : a \in A\}$ is a collection of topological spaces, X is a topological space, and f is a function from X into $\prod_A X_a$. If $\pi_a \circ f$ is continuous for each a in A, then f is continuous.

(b) A one-to-one mapping from a compact space onto a Hausdorff space is a homeomorphism.

(c) In a second countable space, every uncountable set has a limit point.

(d) If K is a sequentially compact subset of a metric space and $\mathcal{G}$ is an open cover of K, then there is a $\delta > 0$ such that every subset of K with diameter less than δ is contained in some member of $\mathcal{G}$.

(4) (10 points each) Prove three of the following.

(a) If f is path homotopic to f', and f' is path homotopic to f'', then f is path homotopic to f''.

(b) Consider the covering map $p : \mathbb{R} \to S^1$ given by $p(t) = (\cos 2\pi t, \sin 2\pi t)$. Define $\varphi : \Pi_1(S^1, (1,0)) \to \mathbb{Z}$ as follows: $\varphi([f]) = \tilde{f}(1)$ where $\tilde{f}$ is the unique lifting of f to a path in $\mathbb{R}$ beginning at 0 (φ is well defined by Theorem 16.5). Then φ is a homomorphism.

(c) If A is a retract of X, then the inclusion map $i : A \to X$ induces an monomorphism $i_* : H_q(A) \to H_q(X)$. (Assume only the Axioms of Homology and their consequences.)

(d) If (X, A) is a topological pair, and the inclusion map $i : A \to X$ is a homotopy equivalence, then $H_q(X, A) \cong 0$ for each q. (Assume only the Axioms of Homology and their consequences.)