Qualifying Exam
Analysis
August 2009

1 Definitions

Give thorough and precise definitions of ALL of the properties below.

(a) measure;

(b) σ-finite measure;

(c) semi-finite measure;

(d) Banach space;

(e) linear functional;

(f) uniformly continuous;

(g) dual of a space;

(h) weak topology;

(i) weak* topology.

2 Questions

Give thorough and precise answers to TWO of the following THREE questions.

i. Let (X, M, μ) be a measure space. What do we mean when we say that μ is:

  (i) Monotonic?
  (ii) Subadditive?
  (iii) Continuous from below?
  (iv) Continuous from above?

ii. Let $(X, \mathcal{M}, \mu)$ be a measure space. How is the integral with respect to μ constructed/defined?

iii. Describe all of the different modes of convergence for sequences of functions that we discussed in the course.

3 Theorems

State ALL of the following theorems. Prove ONE of the following FOUR theorems.

i. The Monotone Convergence Theorem;

ii. Egoroff's Theorem;

iii. Riesz Representation Theorem;

iv. Alaoglu's Theorem.

TURN OVER

4 Problem set I

Solve THREE of the following

Problem 4.1 *Let* μ^* *be an outer measure defined on* X. *Let* $A, B \subseteq X$. *If* $\mu^*(A) = 0$ *then*

$$\mu^*(A \cup B) = \mu^*(B).$$

Problem 4.2 *Show that the inner product in a Hilbert space is continuous.*

Problem 4.3 *Let* $(X, \mathcal{M}, \mu)$ *be a measure space. Show that if* $\mu(E_n) < \infty$ *for each* $n \in \mathbb{N}$ *and* $\chi_{E_n} \to f$ *in* L^1 *then* f *is (a.e. equal to) the characteristic function of a measurable set.*

Problem 4.4 *Prove that a bounded linear operator is uniformly continuous. Also show that if a linear operator is continuous at one point then it is bounded.*

5 Problem set II

Solve ONE of the following

Problem 5.1 *Let* $(X, \mathcal{M}, \mu)$ *be a measure space. Prove that if* $f \in L^+$, *let* $\lambda(E) = \int_E f d\mu$ *for* $E \in \mathcal{M}$. *Then* λ *is a measure on* $\mathcal{M}$ *and for any* $g \in L^+$, $\int g d\lambda = \int f g d\mu$. *(First suppose that* g *is simple.)*

Problem 5.2 *Let* S *be a bounded subset of a normed space* X. *Let* $\mathcal{F}$ *be a set of functionals in* X^* *and let* $\mathcal{F}_0$ *be a dense subset of* $\mathcal{F}$ *(dense in the sense of the norm topology on* X^**). Then* $\mathcal{F}$ *and* $\mathcal{F}_0$ *may generate different weak topologies for* X, *but these topologies are the same on* S.