Topology **Ph.D. Qualifying Exam** **August 18, 2009**

(1) (3 points each) Complete each of the following definitions.

(a) A topological space is *connected* ...

(b) A topological space is *second countable* ...

(c) The *decomposition topology* on a decomposition $\mathcal{D}$ of a topological space X is ...

(d) Suppose $\{X_a : a \in A\}$ is a collection of topological spaces. The *product topology* on $\prod_A X_a$ is ...

(e) A path f in a space X is said to be *path homotopic* to a path f' in X if and only if ...

(f) Suppose x_0 is a point of a topological space X. The *fundamental group* of X relative to the *base point* x_0 is ...

(g) Suppose $p : E \to B$ and $f : X \to B$ are maps. A *lifting* of f is ...

(h) A *singular p-simplex* of a topological space X ...

(i) The *singular chain group* of a nonempty topological space X in dimension p ...

(j) The *singular homology group* in dimension p of a topological space X is ...

(2) (3 points each) Give an example of each of the following, or state that no such example exists. You need not show any work.

(a) a T_1 space that is not Hausdorff

(b) a sequence, $H_1, H_2, H_3, ...$ of subsets of $\mathbb{R}$ such that $\overline{H}_1 \cup \overline{H}_2 \cup \overline{H}_3 \cup ... \neq \overline{H_1 \cup H_2 \cup H_3 \cup ...}$

(c) a countable basis for the usual topology on $\mathbb{R}$

(d) a T_4 space that fails to be metrizable.

(e) a connected set that fails to be locally connected.

(f) a closed and bounded subset of a metric space that fails to be compact.

(g) a space X and a point $x_0 \in X$ such that $\Pi_1(X, x_0)$ is non-abelian.

(h) two non-homeomorphic spaces X and Y and points $x_0 \in X$ and $y_0 \in Y$ such that $\Pi_1(X, x_0)$ and $\Pi_1(Y, y_0)$ are isomorphic.

(i) a space X and two points, x_0 and x_0' of X such that $\Pi_1(X, x_0)$ and $\Pi_1(X, x_0')$ are not isomorphic.

(j) a covering map for S^1 that is not a homeomorphism.

(k) a space X with a path component A and a point a_0 of A such that $\Pi_1(X, a_0)$ and $\Pi_1(A, a_0)$ are not isomorphic.

(l) A singular p-simplex σ such that $(\partial)_{p-1} \circ (\partial)_p(\varphi_\sigma) \neq 0$.

(3) (10 points each) Prove three of the following.

(a) Suppose $G = \{X_a : a \in A\}$ is a collection of topological spaces, X is a topological space, and f is a function from X into $\prod_A X_a$. If $\pi_a \circ f$ is continuous for each a in A, then f is continuous.

(b) Every metric space is T_4.

(c) Every second countable space is separable.

(d) Every sequentially compact subset of a metric space is compact.

(4) (10 points each) Prove three of the following.

(a) Suppose f is a path in a space X from x_0 to x_1 and f' is a path in X from x_1 to x_2. If g and g' are paths such that $f \simeq_p g$ and $f' \simeq_p g'$, then $f * f' \simeq_p g * g'$.

(b) If A is a deformation retract of X, then A and X are homotopy equivalent.

(c) If $f : X \to Y$ is a map, then the chain map $(f_\#)_p$ maps p-cycles in X to p-cycles in Y for each p.

(d) If (X, A) is a topological pair, and the inclusion map $i : A \to X$ is a homotopy equivalence, then $H_q(X, A) \cong 0$ for each q. (Assume only the Axioms of Homology and their consequences.)