Algebra Qualifying Exam
Wednesday, May, 6, 2009:
2:00 - 5:00 p.m.

A. *Write an essay on the fundamental theorem of finitely generated modules over a principal ideal domain. Include a complete statement, definition of terms, and an outline of the proof identifying essential theorems used. Discuss applications to linear transformations and matrices. Please use complete sentences.*

B. *Give precise and complete definitions for* 12 *of the following* 16 *terms. Clearly identify the definitions that you are submitting.*

1. Groups
a. p-Sylow subgroup of a finite group
b. composition series for a finite group
c. nilpotent group
d. normal subgroup of a group

2. Fields
a. separable field extension
b. Galois group of a field extension
c. algebraically closed field
d. the rational canonical form of a matrix over a field

3. Rings
a. principal ideal domain
b. Jacobson radical of a ring
c. prime ideal of a commutative ring
d. left Artinian ring

4. Modules
a. Noetherian
b. injective
c. semi-simple
d. tensor product

C. *Accurately state, without proof,* 5 *of the following* 7 *theorems and give one non-trivial illustrative example of each theorem stated (one example is sufficient even if there are several parts to the statement). Clearly identify the statements of theorems that you are submitting.*

1. Jordan-Hölder theorem
2. Fundamental theorem of Galois theory
3. Wedderburn-Artin theorem
4. Krull-Schmidt-Azumaya theorem

5. Sylow theorems (three)
6. Lasker-Noether primary decomposition theorem
7. Universal mapping property for exterior powers of modules over a commutative ring

D. *Give a complete statement and proof of* 5 *of the following* 7 *theorems. Any assumptions and theorems used in the proof must be clearly stated. Clearly identify the proofs that you are submitting.*

1. Let H be a normal subgroup of a finite group G. Then G is a solvable group if and only if H and G/H are solvable groups.
2. The functor $Hom_R(-,_R M)$ is an exact functor if and only if $_RM$ is an injective R-module.
3. Nakayama's Lemma for finitely generated modules.
4. Let $K \subset F$ be a field extension. Then

$$E = \{a \in F : a \text{ algebraic over } K\}$$

is a subfield of F.
5. Orbit-stabilizer theorem
6. Maschke's theorem
7. A group with p^2-elements, p a prime, is abelian.

E. *Work* 5 *of the following* 7 *problems. Clearly identify the solutions that you are submitting.*

1. List all abelian groups with 60 elements. List invariant factors and elementary divisors for each group.
2. Show that a group G with 80 elements is not a simple group.
3. Find characteristic polynomial, minimal polynomial, invariant factors, elementary divisors, rational canonical form, and Jordan canonical form for the following $\mathbb{Q}$-matrix

$$A = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

4. Find a splitting field and compute the Galois group for $x^3 - 2 \in \mathbb{Q}[x]$. Justify your conclusions.
5. Let G be a cyclic group with 9 elements. Write the group ring $\mathbb{R}G$ as a product of fields.
6. Let $R = \mathbb{Z}[(-3)^{\frac{1}{2}}]$. Which of the following properties are true for R: Euclidean domain, principal ideal domain, unique factorization domain, Dedekind domain, integrally closed. Justify your answers.
7. Let $F = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and N the subgroup of F generated by the elements $(1, 6, 6), (2, 2, 4), (8, 2, 4)$. Find cyclic groups C_i with $F/N = C_1 \oplus ... \oplus C_n$.