A. Write an essay on the fundamental theorem of finitely generated modules over a principal ideal domain. Include a complete statement, definition of terms, and an outline of the proof identifying essential theorems used. Discuss applications to linear transformations and matrices. Please use complete sentences.

B. Give precise and complete definitions for 12 of the following 16 terms. Clearly identify the definitions that you are submitting.

1. Groups
   a. p-Sylow subgroup of a finite group
   b. composition series for a finite group
   c. nilpotent group
   d. normal subgroup of a group

2. Fields
   a. separable field extension
   b. Galois group of a field extension
   c. algebraically closed field
   d. the rational canonical form of a matrix over a field

3. Rings
   a. principal ideal domain
   b. Jacobson radical of a ring
   c. prime ideal of a commutative ring
   d. left Artinian ring

4. Modules
   a. Noetherian
   b. injective
   c. semi-simple
   d. tensor product

C. Accurately state, without proof, 5 of the following 7 theorems and give one non-trivial illustrative example of each theorem stated (one example is sufficient even if there are several parts to the statement). Clearly identify the statements of theorems that you are submitting.

1. Jordan-Hölder theorem
2. Fundamental theorem of Galois theory
3. Wedderburn-Artin theorem
4. Krull-Schmidt-Azumaya theorem
5. Sylow theorems (three)
6. Lasker-Noether primary decomposition theorem
7. Universal mapping property for exterior powers of modules over a commutative ring

D. Give a complete statement and proof of 5 of the following 7 theorems. Any assumptions and theorems used in the proof must be clearly stated. Clearly identify the proofs that you are submitting.

1. Let $H$ be a normal subgroup of a finite group $G$. Then $G$ is a solvable group if and only if $H$ and $G/H$ are solvable groups.
2. The functor $\text{Hom}_R(-, R M)$ is an exact functor if and only if $R M$ is an injective $R$-module.
4. Let $K \subseteq F$ be a field extension. Then
   $$E = \{ a \in F : a \text{ algebraic over } K \}$$
   is a subfield of $F$.
5. Orbit-stabilizer theorem
6. Maschke’s theorem
7. A group with $p^2$-elements, $p$ a prime, is abelian.

E. Work 5 of the following 7 problems. Clearly identify the solutions that you are submitting.

1. List all abelian groups with 60 elements. List invariant factors and elementary divisors for each group.
2. Show that a group $G$ with 80 elements is not a simple group.
3. Find characteristic polynomial, minimal polynomial, invariant factors, elementary divisors, rational canonical form, and Jordan canonical form for the following $\mathbb{Q}$-matrix
   $$A = \begin{pmatrix}
   0 & 3 & 0 & 0 \\
   1 & 2 & 0 & 0 \\
   0 & 0 & 0 & 0 \\
   0 & 0 & 2 & 0
   \end{pmatrix}$$
4. Find a splitting field and compute the Galois group for $x^3 - 2 \in \mathbb{Q}[x]$. Justify your conclusions.
5. Let $G$ be a cyclic group with 9 elements. Write the group ring $\mathbb{R}G$ as a product of fields.
6. Let $R = \mathbb{Z}[(3)^{\frac{1}{2}}]$. Which of the following properties are true for $R$: Euclidean domain, principal ideal domain, unique factorization domain, Dedekind domain, integrally closed. Justify your answers.
7. Let $F = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $N$ the subgroup of $F$ generated by the elements $(1, 6, 6), (2, 2, 4), (8, 2, 4)$. Find cyclic groups $C_i$ with $F/N = C_1 \oplus \ldots \oplus C_n$. 

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