TOPOLOGY QUALIFYING EXAM

August 2008

Each of the theorems in Part One comes from one of the lists of theorems that we compiled in Math 5330 and Math 5331 last year. For each theorem in Part One, your proof may appeal to any theorem from those lists that precedes it. To prove the theorems in Part Two, you may appeal to any theorem from those lists. No other resources are allowed.

1. Part One

Theorem 1.1. *Suppose K is a subset of a metric space X. If every infinite subset of K has a limit point in K, then K is compact.*

Theorem 1.2. *Suppose $p : E \to B$ is a covering map and $p(e_0) = b_0$. Any path $f : [0,1] \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f} : [0,1] \to E$ beginning at e_0.*

Theorem 1.3. *$\pi_1(X \times Y, x_0 \times y_0)$ is isomorphic with $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

2. Part Two

Definition. Suppose $(X, \mathcal{T})$ is a topological space, and suppose $\mathcal{D}$ is a collection of mutually exclusive subsets of X whose union is X (i.e. a partition of X). We wish to define a new topological space whose points are the members of $\mathcal{D}$. To that end we define a topology $\mathcal{T}_\mathcal{D}$ on $\mathcal{D}$ as follows: a subset $\mathcal{F}$ of $\mathcal{D}$ belongs to $\mathcal{T}_\mathcal{D}$ if and only if $\mathcal{F}^*$ belongs to $\mathcal{T}$.

Theorem 2.1. *$\mathcal{T}_\mathcal{D}$ is a topology for $\mathcal{D}$.*

Definition. For each $x \in X$, $\pi(x)$ is defined to be the member of $\mathcal{D}$ to which the point x belongs.

Theorem 2.2. *$\mathcal{T}_\mathcal{D}$ is the quotient topology on $\mathcal{D}$ induced by π. Thus π is continuous.*

Definition. An open set V in X is said to be saturated with respect to $\mathcal{D}$ if and only if it is the union of members of $\mathcal{D}$ or, equivalently, $V = \pi^{-1}(W)$ for some $W \in \mathcal{T}_\mathcal{D}$. $\mathcal{D}$ is said to be *usc* if and only if, for each $F \in \mathcal{D}$ and each open subset U of X containing F, there is a saturated open subset V of X such that $F \subset V \subset U$.

Theorem 2.3. *$\mathcal{D}$ is usc if and only if π is closed.*

Theorem 2.4. *If X is a T_1-space and $\mathcal{D}$ is usc, then each member of $\mathcal{D}$ is a closed subset of X.*

Theorem 2.5. *If X is a compact metric space and $\mathcal{D}$ is usc, then $(\mathcal{D}, \mathcal{T}_\mathcal{D})$ is a compact metrizable space. (Hint: by Theorem 9.5, it suffices to show that $(\mathcal{D}, \mathcal{T}_\mathcal{D})$ is a compact Hausdorff space.)*