**Topology Qualifying Examination**

**Instructions:** Work two of the problems from each section

# 1 Topology Section

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<u>PROBLEM</u> **1 Local-Compactness.**

**Theorem 1.1** Let $X$ be a topological space and let $A \subseteq X$ be dense. If $U \subseteq X$ is open then $\overline{(A \cap U)} \supseteq U$.

**Definition 1.1** Let $X$ be a Hausdorff space, and let $x \in X$. We say that $X$ is *locally compact at* $x$ if for each $U \ni x$ open there is an open set $x \in V \subseteq U$ such that $\overline{V} \subseteq U$ is compact.

**Theorem 1.2** $\mathbb{Q}$ is not locally compact in $\mathbb{R}$ with its usual topology.

**Theorem 1.3** Let $X_\alpha$ be a topological space for each $\alpha \in A$. Then $\prod_{\alpha \in A} X_\alpha$ is locally compact if, and only if, each $X_\alpha$ is locally compact and $X_\alpha$ is compact for all but finitely many $\alpha \in A$.

**Theorem 1.4** If $A$ is a dense locally compact subspace of a Hausdorff space $X$ then $A$ is open in $X$.

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<u>PROBLEM</u> **2 Upper Semi-Continuity & Nets.**

**Definition 2.1** Let $f : X \to \mathbb{R}$ be a function with $X$ a topological space. $f$ is *upper semi-continuous* if the set $\{x | f(x) \geq a\}$ is closed for each real number $a$.

**Theorem 2.1** If $f$ and $g$ are upper semi-continuous and $t$ is a non-negative real number then $f + g$ and $tf$ are upper semi-continuous.

**Definition 2.2** The *upper topology*, $\mathcal{U}$, on $\mathbb{R}$ consists of the empty set and all sets of the form $\{t : t < a\}$ for all $a \in \mathbb{R}$.

**Theorem 2.2** Let $f : X \to \mathbb{R}$ be a function on the topological space $X$. Then $f$ is upper semi-continuous if and only if it is continuous with respect to the upper topology $\mathcal{U}$.

**Definition 2.3** Let $D$ be a directed set. If $\{S_n | n \in D\}$ is a net of real numbers then $\limsup\{S_n | n \in D\}$ is defined to be $\lim\{\sup\{S_m | m \in D \text{ and } m \geq n\} | n \in D\}$ where this limit is taken relative to the usual topology on $\mathbb{R}$.

**Theorem 2.3** A net $\{S_n | n \in D\}$ of real numbers converges to $s$ relative to $\mathcal{U}$ if and only if $\limsup\{S_n | n \in D\} \leq s$.

**Theorem 2.4** Let $f : X \to \mathbb{R}$ be a function on the topological space $X$. Then $f$ is upper semi-continuous if, and only if $\limsup\{f(x_n) | n \in D\} \leq f(x)$ whenever $\{x_n | n \in D\}$ is a net in $X$ converging to a point $x$.

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## PROBLEM 3 Connection im kleinen & Sierpinski's Property S

**Definition 3.1** Let $X$ be a topological space, and let $x \in X$. We say that $X$ is *connected im kleinen at* $x$ if each open set $U \ni x$ contains an open set $V \ni x$ such that any pair of points in $V$ lie in some connected subset of $U$.

**Theorem 3.1** If $X$ is locally connected at $x$ then it is connected im kleinen at $x$.

**Note:** Let $B$ denote the *infinite broom space* pictured on page 163 of Munkres.

**Theorem 3.2** Being connected im kleinen at $x$ does not imply being locally connected at $x$.

**Theorem 3.3** If $X$ is connected im kleinen at each point $x \in X$ then $X$ is locally connected.

**Definition 3.2** A space $X$ has *Property S* if every open cover of $X$ can be refined by a cover consisting of a finite number of connected sets. (This property was introduced by Sierpinski in 1920.)

**Theorem 3.4** If $X$ has Property S then $X$ is connected im kleinen at each point.

**Theorem 3.5** A compact Hausdorff space is locally connected if, and only if, it has Property S.

**Theorem 3.6** Not every locally connected Hausdorff space has Property S.

## 2 Algebraic Topology Section

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**PROBLEM 4 Homology I.**

**Theorem 4.1** (Using only the Eilenberg-Steenrod axioms and assuming all necessary pairs are admissible, prove the following.) If $A$, $B$ and $X$ are spaces such that $B \subset A \subset X$, and there is a deformation retraction of $X$ onto $A$, then $H_p(X, B) \approx H_p(A, B)$ for all $p$.

**Question 4.1** *If a simplicial complex $K$ is the union of two connected subcomplexes $K_0$ and $K_1$ such that $|K_0| \cap |K_1|$ consists of two points, what can be said about the homology of $K$?*

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**PROBLEM 5 Covering Spaces.**

**Theorem 5.1** Let $p : \tilde{X} \to X$ be a covering space, and let $A \subseteq X$. Let $\tilde{A} = p^{-1}(A)$. Then the restriction $p : \tilde{A} \to A$ is a covering space.

**Theorem 5.2** Let $p_1 : \tilde{X_1} \to X_1$ and $p_2 : \tilde{X_2} \to X_2$ be covering spaces. Then their product is also a covering space.

**Theorem 5.3** Let $p : \tilde{X} \to X$ be a covering space with $p^{-1}(x)$ finite and nonempty for all $x \in X$. Then $\tilde{X}$ is compact Hausdorff if, and only if, $X$ is compact Hausdorff.

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**PROBLEM 6 Homology II.**

**Question 6.1** *Compute the homology groups of the $\Delta$-complex, $X$, obtained from $\Delta^n$ by identifying all faces of the same dimension. (Notice $X$ has a single $k$-simplex for each $k \leq n$.)*

**Theorem 6.1** Let $X$ be a space and $A \subseteq X$. Then $H_0(X, A) = 0$ if, and only if, $A$ meets each path component of $X$.

**Theorem 6.2** Let $X$ be a space and $A \subseteq X$. Then $H_1(X, A) = 0$ if, and only if, $H_1(A) \xrightarrow{i_*} H_1(X)$ is surjective and each path component of $X$ contains at most one path-component of $A$.

**Theorem 6.3** Let $X = [0, 1]$, and let $A = \{\frac{1}{n}\}_{n \in \mathbb{N}} \cup \{0\}$. Then $H_1(X, A)$ is not isomorphic to $H_1(X/A)$.

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**PROBLEM 7 Homotopy**

**Theorem 7.1** Let $X_0$ be the path-component of a space $X$ containing the basepoint $x_0$. Then inclusion $X_0 \hookrightarrow X$ induces an isomorphism $\pi_1(X_0, x) \to \pi_1(X, x_0)$.

**Theorem 7.2** Let $X$ be a space and $A$ a path-connected subspace containing the basepoint $x_0$. Then the homomorphism $\pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by inclusion $A \hookrightarrow X$ is surjective if, and only if, every path with endpoints in $A$ is homotopic to a path in $A$.