TOPOLOGY QUALIFYING EXAM

August 2005

Problem 1. Using only the Eilenberg-Steenrod axioms and assuming all necessary pairs are admissible, prove the following. If A is a deformation retraction of X, then $H_p(X, A)$ is trivial for all p.

Problem 2. If a simplicial complex K is the union of two connected acyclic subcomplexes K_0 and K_1, what can be said about the homology of K?

Problem 3. If D is an open subset of a metric space, then, for each point x of D, there is an $\epsilon > 0$ such that $B(x, \epsilon) \subset D$.

Problem 4. Every sequence in a compact metric space has a convergent subsequence.

Problem 5. If H and K are mutually separated sets in a topological space and $H \cup K$ is open, then each of H and K is open.

Definitions. A topological space X will be said to have *Property* P provided every point x of X is a limit point of each component of $X - \{x\}$. A *nonseparating point* of a topological space is a point x whose complement is connected.

Problem 6. Notice that $\{x \in \mathbb{R} : x \geq 0\}$ has property P and only 1 nonseparating point. The following sequence of results shows that this is not possible for compact Hausdorff spaces.

(1) If x is a point of a topological space X with property P, and $X - \{x\}$ is the union of two mutually separated sets, H and K, then $H \cup \{x\}$ is a closed connected subset of X.

(2) Suppose X has property P, and suppose p is a point of X such that, for each point x different from p, $X - \{x\}$ is the union of two mutually separated sets H_x and K_x, with p in H_x.
   (a) If $a, b \neq p$, $b \notin H_a$, then $H_a \cup \{a\} \subset H_b$.
   (b) The collection $\mathcal{C} = \{H_x : x \in X \text{ and } x \neq p\}$ is a collection of proper open subsets of X with a monotonic subcollection that covers X.

(3) Every compact Hausdorff space with property P has at least two nonseparating points.

Remark. In a compact Hausdorff space, property P is equivalent to connectedness. Hence (7.3) implies that every compact connected Hausdorff space has at least two non-separating points. It can be shown that every compact connected metrizable space with only two nonseparating points is homeomorphic to $[0, 1]$.