QUALIFYING EXAM IN REAL VARIABLES
BAYLOR UNIVERSITY
SPRING 2007

Complete four of the following six questions.

1. State precisely the following:
   a) Fatou’s Lemma
   b) The Lebesgue Monotone Convergence Theorem
   c) The Hahn-Banach Theorem
   d) The Fubini Theorem
   e) The Closed Graph Theorem
   f) The Krein-Milman Theorem
   g) The Riesz Representation Theorem
   h) The Radon-Nikodym Theorem

2. Let $f \in L^1[0,1]$. Must
   \[ \lim_{\alpha \to \infty} \alpha \cdot |\{x : |f(x)| > \alpha\}| = 0 \quad ? \]
   Justify your answer.

3. What is the Cantor-Lebesgue function? How may it be used to construct a set $S$ which is Lebesgue measurable but not a Borel set?

4. Let $f_n(t) = \sin nt \quad (-\pi \leq t \leq \pi)$. Show that if $1 \leq p < \infty$ then $f_n \rightharpoonup 0$ weakly in $L^p(-\pi, \pi)$ but not strongly in $L^p(-\pi, \pi)$. 
5.  a) Is \( \mathbb{Q} \) a \( G_\delta \)? Justify your answer.

b) Is there a real-valued function on \( \mathbb{R} \) which is continuous on \( \mathbb{Q} \) but discontinuous on \( \mathbb{R} - \mathbb{Q} \)? Justify your answer.

6.  a) Let \( a, b \) be nonnegative numbers, and \( p, q \) such that \( 1 < p < \infty \) and \( 1/p + 1/q = 1 \). Establish Young’s inequality:

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}.
\]

b) Using Young’s Inequality prove the Hölder inequality: If \( f \in L^p[0,1] \) and \( g \in L^q[0,1] \), where \( p \) and \( q \) are as above, then \( fg \in L^1[0,1] \) and

\[
\int |fg| \leq \|f\|_p \cdot \|g\|_q.
\]

c) For \( 1 < p < \infty \) and \( g \in L^q \), consider the linear functional \( F \) on \( L^p \) given by

\[
F(f) = \int fg.
\]

Show that \( \|F\| = \|g\|_q \).