QUALIFYING EXAM IN REAL VARIABLES
BAYLOR UNIVERSITY
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Complete four of the following six questions.

1. State precisely the following:

a) Fatou's Lemma

b) The Lebesgue Dominated Convergence Theorem

c) The Hahn-Banach Theorem

d) The Fubini Theorem

e) The Uniform Boundedness Principle

f) The Krein-Milman Theorem

g) Alaoglu's Theorem

h) The Radon-Nikodym Theorem

2. Define the function f on the interval $(0,1)$ as follows. If $x = x_1 x_2 x_3 \ldots$ is the unique nonterminating decimal expansion of $x \in (0,1)$, define $f(x) = \max_n\{x_n\}$. Prove that f is measurable.

3. Let C be the space of all real continuous functions on $[0,1]$ with the supremum norm. Let X_n be the subset of C consisting of those f for which there exists a $t \in [0,1]$ such that $|f(s) - f(t)| \leq n|s-t|$ for all $s \in [0,1]$. Fix n and prove that each open set in C contains an open set which does not intersect X_n. Show that this implies the existence of a dense G_δ in C which consists entirely of nowhere differentiable functions.

4. Does there exist a continuous function f on $[0,1]$ such that

$$\int_0^1 x^n f(x)\, dx = \begin{cases} 1 & n = 1 \\ 0 & n = 2, 3, 4, \ldots \end{cases} \qquad ?$$

5. Let $\ell^\infty(\mathbb{R})$ denote the space of bounded real sequences $\{x_n\}$, $n = 1, 2, 3, \ldots$. Show there exists a continuous linear functional $L \in \ell^\infty(\mathbb{R})^*$ with the following properties:

i) $\inf_n x_n \leq L(\{x_n\}) \leq \sup_n x_n$
ii) If $\lim_{n\to\infty} x_n = a$ then $L(\{x_n\}) = a$
iii) $L(\{x_n\} = L(\{x_{n+1}\})$.

Hint: consider $V \subset \ell^\infty(\mathbb{R})$ generated by the sequences $\{x_{n+1} - x_n\}$. Show $\{1, 1, 1, \ldots\} \notin \bar{V}$ and apply the Hahn-Banach Theorem.

6. Let H be an infinite dimensional Hilbert space. A linear operator $T : H \to H$ is said to be *compact* if and only if the closure of

$$T(B) = \{g \in H : g = Tf \text{ for some } f \in H \text{ with } \|f\| \leq 1\}$$

is compact. T is said to be of *finite rank* if and only if its range is finite dimensional.

a) Is the identity operator on H compact?

b) Show that if $\{T_n\}$ is a family of compact linear operators with $\|T_n - T\| \to 0$ as $n \to \infty$ then T is compact. [Hint: diagonalization, $\epsilon/3$.]

c) Show that if T is compact there exists a sequence T_n of operators of finite rank such that $\|T_n - T\| \to 0$. [Hint: what is special about Hilbert spaces?]