2005 REAL ANALYSIS QUALIFYING EXAMINATION

Examining Committee

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Write your final solution to each problem on the official problem sheet for that problem. Place the problem sheets in order with this the ID code sheet on top. Your name should not be written on any part of this exam.

Throughout, assume $(\mathbb{X}, \mathcal{M}, \mu)$ is a measure space. We will use m to denote Lebesgue measure. Make no other assumptions about the given spaces unless told to do so.

PART I: Required Problems. Complete each of the following.

1. Consider $\chi_{\mathbb{K}\cap[0,1]}$ where $\mathbb{K}$ is the Cantor set.
   (a) Is this function Riemann integrable on $[0,1]$? Explain why or why not.
   (b) Is this function Lebesgue integrable on $[0,1]$? Explain why or why not.
2. Suppose $\{f_n\}_{n=1}^{\infty} \subset L^1(\mu)$ and $f_n \to f$ uniformly on $\mathbb{X}$.
   (a) Show that if $\mu(\mathbb{X}) < \infty$, then $f \in L^1(\mu)$ and $\int_{\mathbb{X}} f_n \, d\mu \to \int_{\mathbb{X}} f \, d\mu$.
   (b) Show that if $\mu(\mathbb{X}) = \infty$, then the conclusions in (a) can fail. (Provide a counterexample on $\mathbb{R}$ with Lebesgue measure.)
3. Prove that a function F is an indefinite integral if and only if it is absolutely continuous.
4. Let A and B be Lebesgue measurable subsets of $\mathbb{R}$. Define $\mu(E) = 2m(E \cap A) + m(E \cap B)$.
   (a) Show that $\mu \ll m$.
   (b) Find the Radon-Nikodym derivative $d\mu/dm$.
5. Let $\mathbb{X} = \mathbb{Y} = [0,1]$, μ be Lebesgue measure on $[0,1]$, and ν be the counting measure on $[0,1]$. Consider the diagonal $D = \{(x,x) : x \in \mathbb{X}\}$ of the product space $\mathbb{X} \times \mathbb{Y}$.
   (a) Show that D is a measurable subset of $\mathbb{X} \times \mathbb{Y}$.
   (b) Show that both of the iterated integrals exist (compute them)

$$\int_{\mathbb{X}} \int_{\mathbb{Y}} \chi_D \, d\nu \, d\mu, \qquad \int_{\mathbb{Y}} \int_{\mathbb{X}} \chi_D \, d\mu \, d\nu.$$

   (c) Show that χ_D is *not* $\mu \times \nu$-integrable. Does this contradict the theorems of Fubini/Tonelli?

PART II: Option Problems. Choose FIVE of the following TEN problems.

6. (a) Suppose $\{E_i\}_{i=1}^{\infty}$ is a sequence of measurable sets such that $E_{i+1} \subset E_i$ and $m(E_1) < \infty$. Show that

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n).$$

(b) Show that the condition $m(E_1) < \infty$ is necessary by giving an example of a decreasing sequence of measurable sets $\{E_i\}$ such that $\cap E_i = \emptyset$ while $m(E_i) = \infty$ for all i.

7. Consider $(\mathbb{X}, \mathcal{S}, \mu)$. Suppose $f \in L^1(\mu)$ and let $\nu(E) = \int_E f\, d\mu$, $E \subset \mathcal{S}$.
   (a) Show that ν is a signed measure.
   (b) Find a Hahn decomposition of $\mathbb{X}$ with respect to ν.
   (c) Find the Jordan decomposition of ν.

8. Let f be Lebesgue integrable on $\mathbb{R}$. The *Fourier transform of f*, denoted by $\mathcal{F}$, is given by

$$\mathcal{F}(t) = \int_{\mathbb{R}} e^{itx} f(x)\, dx = \int_{\mathbb{R}} \cos(tx) f(x)\, dx + i \int_{\mathbb{R}} \sin(tx) f(x)\, dx.$$

Show that $\mathcal{F}$ is continuous and bounded on $\mathbb{R}$.

9. Show that for a linear operator, continuity (at a point) and boundedness (on the space) are equivalent.

10. State and prove the Lebesgue Dominated Convergence Theorem on $(\mathbb{X}, \mathcal{M}, \mu)$.

11. (a) Prove that if $f(x)$ is a measurable function on the set E, then $S_r = \{x \in E : f(x) = r\}$ is a measurable set for each $r \in \mathbb{R}$.
    (b) Show that the converse is false: even if S_r is a measurable set for each $r \in \mathbb{R}$, f can be a nonmeasurable function on E.

12. (a) Compare and contrast the following three decompositions: Hahn, Jordan, and Lebesgue.
    (b) Interpret the Riesz Representation Theorem in the $L^2(\mu)$ setting. (Hint: $p = 2$ is special.)

13. Suppose f and g are Lebesgue measurable on $\mathbb{R}$. The *convolution* of f and g (denoted by $f * g$) is given by

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y) g(y)\, dy,$$

provided the integral exists (the integration here is with respect to Lebesgue measure). Show that if $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then $f * g \in L^1(\mathbb{R})$ (in particular, $f * g$ is well-defined and finite a.e. with respect to Lebesgue measure). (Hint: Use Fubini's Theorem along the way.)

14. Consider $C[0,1]$ with the two norms

$$||f||_\infty = \sup_{x\in[0,1]} |f(x)| \qquad \text{and} \qquad ||f||_1 = \int_0^1 |f(x)|\, dx.$$

Show that the identity operator $I : (C[0,1], ||\cdot||_\infty) \to (C[0,1], ||\cdot||_1)$ is continuous, onto, but *not* open. Does this contradict the Open Mapping Theorem? Explain.

15. (a) Suppose f is Lebesgue integrable on $[a,b]$. Show that

$$F(x) = \int_a^x f(t)\, dm(t)$$

is a continuous function of bounded variation on $[a,b]$.

(b) What inference can you make if $F(x) \equiv 0$ on $[a,b]$? Prove your claim.