MTH 5310, 5311 Qualifying Examination May 1, 2004

The following notations are used: $\mathcal{M}$ =Lebesgue measurable subsets of $\mathbb{R}$, m^* = Lebesgue outer measure, m = Lebesgue measure, $\int$ =Lebesgue integration, $\ll$ denotes absolute continuity of one measure w.r.t. another measure.

I. Select three from 1, 2, 3 and 4.

1(a). Let $A \subseteq \mathbb{R}$. Prove there exists $G \in \mathcal{G}_\delta$ such that $A \subseteq G$ and $m^*A = m^*G$.

(b). Let A and G be as in (a) but with $A \notin \mathcal{M}$. Prove that $m^*(G \setminus A) > 0$.

2(a). Let $E_1 \supseteq E_2 \supseteq \cdots$, where $E_n \in \mathcal{M}$, for each n. Suppose that for some $k \in \mathbb{N}$, $m(E_k) < \infty$. Prove

$$m\left(\cap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m\left(E_n\right).$$

(b). Show that $m\left(E_k\right) < \infty$, for some $k \in \mathbb{N}$, is a necessary condition in (a).

3. Assume $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable and $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable. Prove that $f \circ g : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable.

4. Assume the measure space $(X, \mathcal{A}, \mu)$ is complete. If f is μ-measurable and $f = g$ a. e., show that g is μ-measurable.

II. Select three from 5, 6, 7, 8 and 9.

5. Let $\langle f_n \rangle$ be a sequence of nonnegative measurable functions on a set $E \in \mathcal{M}$ and assume $f_n \to f$ on E. If $f_n \leq f$, for each n, prove that

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

6. Show that $L^\infty[0,1] \subset \cap_{p\geq 1} L^p[0,1]$; (be sure to show the inclusion is also proper inclusion).

7. Let $\langle f_n \rangle$ be a sequence of nonnegative Lebesgue measurable functions that decrease pointwise to f. If $\int f_1 < \infty$, show that

$$\lim_{n\to\infty} \int f_n = \int f.$$

8. Let $\langle f_n \rangle$ be a sequence of real-valued measurable functions with domain $E \in \mathcal{M}$, where $m(E) < \infty$. Draw arrows giving the positive implications between the designated types of convergence. Also, give counterexamples for all the negative implications.

   (a) uniform convergence (b) almost uniform convergence

   (c) convergence in measure (d) a. e. convergence

9. Let $h : \mathbb{R} \to \mathbb{R}$ be continuous and bounded. If $f \in L^1(\mathbb{R})$, show that g defined by

$$g(x) = \int_{\mathbb{R}} h(x-y)f(y)dy$$

   is continuous and bounded on $\mathbb{R}$.

III. Select <u>three</u> from 10, 11, 12 and 13.

10. Define

$$g(x) = \begin{cases} \frac{(-1)^{n+1}}{n^2}, & x \in [n, n+1),\ n = 1, 2, \ldots, \\ 0, & x < 1, \end{cases}$$

   and set $\nu(E) = \int_E g\,dm = \int_E g(x)dx$, for all $E \in \mathcal{M}$. Now, $(\mathbb{R}, \mathcal{M}, \nu)$ is a signed measure space.

   (a). Show $\nu \ll m$, (i.e., $|\nu| \ll m$).
   (b). Show that $(\mathbb{R}, \mathcal{M}, \nu)$ is not complete.
   (c). Give a Hahn decomposition w.r.t. ν of $\mathbb{R}$.
   (d). Give a Jordan decomposition of ν.

11. Let $A = \mathbb{Q} \cap [0,1] = \{r_1, r_2, \ldots\}$ be an enumeration of the rationals in $[0,1]$. Define

$$F(x) = \begin{cases} 0, & x < 0, \\ \sum_{r_n \le x} \left(\frac{1}{2^n}\right), & x \ge 0. \end{cases}$$

   (a). Determine the associated Borel measure μ_F; that is, determine $\mu_F(E)$, for each Borel set E.
   (b). Show $\mu_F \not\ll m$.
   (c). Let $\phi(x) = x$. Then evaluate $\int_0^1 \phi\, dF$.

12. Suppose $f : (0,1) \to \mathbb{R}$ is Lebesgue integrable on $(0,1)$. Define

$$g(x) = \int_x^1 t^{-1} f(t)dt.$$

Show g is integrable on $(0,1)$ and that

$$\int_0^1 g(x)dx = \int_0^1 f(x)dx.$$

13. Suppose $\langle f_n \rangle$ is a sequence of measurable functions so that $f_n \to f$ in measure and $|f_n| \leq g$, for all n, where g is an integrable function. Prove that

$$\lim_{n\to\infty} \int f_n = \int f.$$

IV. Select <u>two</u> from 14, 15, 16 and 17.

14. State and prove Fatou's Lemma.
15. State and prove the Lebesgue Dominated Convergence Theorem.
16. State and prove Egoroff's Theorem.
17. State and prove the Jordan Decomposition Theorem.