Algebra Qualifier Exam
July 30, 2008

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(A) Definitions:

(A.1) Define what it means that the group G acts on the set Xt.
(A.2) A subgroup S of the finite group G is a p-Sylow subgroup, if...
(A.3) The subgroup S of the group G is characteristic, if...
(A.4) The ring R is an integral domain, if...
(A.5) The ring R is a PID, if...
(A.6) The ring R is a UFD, if...
(A.7) Two elements a, b of the ring R are associates, if...
(A.8) The complex number u is an algebraic integer, if...
(A.9) The element s of the ring R is irreducible, if...
(A.10) The field E is the splitting field of the polynomial $f(x) \in k[x]$, if...
(A.11) $Gal(E \mid k) = ...$
(A.12) The field E is a Galois extension of the field k, if...
(A.13) The field E is a radical extension of the field k, if..
(A.14) The subgroup G of S_X operates transitively on the set X, if...
(A.15) The ring R is Noetherian, if...
(A.16) The sequence $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ is short exact, if...
(A.17) The (left) R-module M is simple, if...
(A.18) The (left) R-module M is injective, if...
(A.19) The Jacobson radical $J(R)$ of the ring R is $J(R) = ...$
(A.20) The ring R is semisimple, if...
(A.21) If A is a right and B a left R-module the the tensor product $A \otimes_R B$ is defined as....
(A.22) The integral domain R is integrally closed, if...
(A.23) The (left) R-module M has a composition series, if...
(A.24) If $f(x) = \sum_{i=0}^{n} a_i x^i \in k[x]$ is a monic polynomial, then the companion matrix $C(f(x)$ is...
(A.25) The matric J is a Jordan block matrix with eigenvalue λ, if...

(B) State the indicated result:

(B.1) The Orbit-Stabilizer Theorem.
(B.2) The First Isomorphism Theorem for groups.
(B.3) State the Eisenstein Criterion.
(B.4) State the Fundamental Theorem of Galois Theory.
(B.5) State Hilbert's Basis Theorem.
(B.6) State Zorn's Lemma.
(B.7) State the Abel-Ruffini Theorem: If $n \geq 5$, then...
(B.8) State Steinitz' Theorem on field extensions with only finite many intermediate fields.
(B.9) The Third Isomorphism Theorem for R-modules.
(B.10) If A_i $(i \in I)$ and B are R-modules, then $Hom_R(\bigoplus_{i \in I} A_i, B) \cong ...$
(B.11) State Schanuel's Lemma.
(B.12) State Baer's Criterion for injective modules.
(B.13) State Nakayama's Lemma.
(B.14) The ring R is semisimple $\iff$ ...
(B.15) State Maschke's Theorem for group rings.
(B.16) State the Artin-Wedderburn Theorem.
(B.17) If R is a UFD, then R is .... in its field of fractions.
(B.18) If R is a PID and M a submodule of a free R -module, then M is...

(B.19) Let E be an algebraic number field such that its ring $\mathbf{O}_E$ of algebraic integers is a UFD. Then $\mathbf{O}_E$ is actually a ...

(B.20) State Dirichlet's Theorem on $U(\mathbf{O}_E)$, the group of units of $\mathbf{O}_E$.

(C) Prove the following:

(C.1) If $\varphi : G \to H$ is a homomorphism of groups, then $\ker(\varphi)$ is a normal subgroup of G.

(C.2) If R is a PID, then R is a UFD.

(C.3) Let G be a finite, non-abelian, simple group of order $p^e m$ such that p is a prime not dividing m. Show that p divides $(m-1)!$.

(C.4) If G is a finite group with a normal p-Sylow subgroup S, then all elements of order p in G are actually in S.

(C.5) If J is a maximal ideal of the commutative ring R, then R/J is a field.

(C.6) Give a proof of the Eisenstein Criterion.

(C.7) If R is a Noetherian ring and J an ideal of R, then R/J is Noetherian.

(C.8) Use Zorn's Lemma to prove: If $1 \in R$ is a commutative ring, then R has a maximal ideal.

(C.9) Let $p(x) \in k[x]$ be an irreducible polynomial of degree n and E a finite dimensional field extension of k such that E contains a root of $p(x)$. Show that n divides $[E : k]$.

(C.10) Let $f(x) \in k[x]$ and E a field extension of k. Let $\gamma \in Gal(E \mid k)$. Show that γ induces a permutation of the set $\{u \in E : f(u) = 0\}$.

(C11) Let E be a Galois extension of the field k and $f(x) \in k[x]$ a polynomial that has a root in E. Show that $f(x)$ splits over E

(C12) Use Zorn's lemma to show that every vector space has a basis.

(C.13) Baer's Criterion (for injective modules).

(C.14) Let R be a ring and L, K minimal ideals of R. show that either $LK = \{0\}$ or $K \cong L$ as R-modules.

(C.15) Maschke's Theorem.

(C.16) If R is a PID and M a divisible R-module, then M is injective.

(C.17) Let A be a S-R-bimodule and B a left R-module. Show that $A \otimes_R B$ is a left S-module.

(C.18) Let E be an algebraic number field and $\mathbf{O}_E$ its ring of algebraic integers. Prove that $\mathbf{O}_E$ is a Noetherian ring.

(C.19) Same notation as in (C.17). Prove that for any $u \in E$ there is some natural number $n \geq 1$ such that $nu \in \mathbf{O}_E$.

(D) Work the following problems:

(D.1) Show: If G is a group of order 20, then G has a normal 5-Sylow subgroup.

(D.2) Show that the commutator subgroup G' of the group G is a fully invariant subgroup of G.

(D.3) Show that $f(x) = 5x^9 + 6x^5 - 3x + 12$ is irreducible over the field Q of rational numbers.

(D.4) For $u = \sqrt{1 - \sqrt{3}}$ find the irreducible polynomial of u. Is u an algebraic integer? Why?

(D.5) Explain why $E = Q(i, \sqrt{2})$ is the splitting field of $f(x) = (x^2 - 2)(x^2 + 1)$.

(D.6) Same notations as in (D.5). Explain why E is a Galois extension of Q.

(D.7) Let E be an extension of the field k such that $[E : k]$ is a prime number. Explain why there is no proper intermediate field.

(D.8) Let R be a commutative ring and M a left R-module. Show that R and $Hom_R(R, M)$ are isomorphic as R-modules.

(D.9) Explain why Z, the ring of integers, is Noetherian but not Artinian and why $J(Z) = \{0\}$.

(D10) Let k be a field of characteristic $p > 0$ and G a finite group such that p divides $|G|$. Prove that kG is not semisimple.

(D.11) Let T be a torsion Abelian group. Show that $Q \otimes_Z T = \{0\}$.

(D.12) Let M be a 4×4 real matrix with characteristic polynomial $(x-2)^2(x-3)^2$ and minimal poynomial $(x-2)(x-3)^2$. Find the Jordan normal form of M.

(D13) Let E be an algebraic number field and $\mathbf{O}_E$ its ring of algebraic integers. Let P be a prime ideal of $\mathbf{O}_E$. Show that there is exactly one prime number p such that $p \in P$.