Algebra Qualifying Exam May 17, 2007

Part A – Definitions

Give the definitions of the following terms:

1. (a) Normal subgroup
   (b) Simple group
   (c) Composition series of a group
   (d) Solvable group
2. (a) Degree of a field extension
   (b) Algebraic extension
   (c) Separable extension
   (d) Galois extension
3. (a) Prime ideal (context: commutative rings)
   (b) Maximal ideal
   (b) Radical of an ideal
   (d) Noetherian ring
4. (a) Free module
   (b) Projective module
   (c) Projective resolution of a module
5. (a) Simple module (context: noncommutative rings)
   (b) Semisimple module
   (c) Simple ring
   (d) Semisimple ring

Part B – Theorems

Give the statements of the following theorems:

1. Sylow Theorems
2. Fundamental Theorem of Galois Theory
3. Cayley-Hamilton Theorem
4. Wedderburn-Artin Theorem
5. Schur's Orthogonality Relations

Part C – Proofs of Theorems

1. Prove the first Sylow Theorem.
2. Use Kronecker's Theorem to prove that for every prime power $q = p^n$ there exists a field of order q.
3. Prove Hilbert's Basis Theorem.
4. Prove the existence of the connecting homomorphism δ of the Snake Lemma.
5. Sketch a proof of Jacobson's Density Theorem.

Part D – Computational Problems

Show work to justify your answers.

1. How many abelian groups (up to isomorphism) are there of order 48?
2. (a) How many Sylow 5-subgroups does a simple group of order 60 have?
   (b) How many Sylow 3-subgroups does a simple group of order 60 have?
3. (a) Compute the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Z}$.
   (b) Compute the Galois group of $\mathbb{Q}[\sqrt{2} + \sqrt{3}]$ over $\mathbb{Q}$.
4. Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if $\gcd(m, n) = 1$.
5. Compute the character table of the symmetric group S_3.

Part E – Theoretical Problems

1. Let G be a nonabelian group of order p^3, where p is a prime number. Prove that the center of G has order p.
2. Let A be a commutative ring and I an ideal of A. Prove that A/I is a field if and only if I is a maximal ideal of A.
3. If $f(x) \in k[x]$ is a separable polynomial, prove that its discriminant lies in k.
4. Let A be a commutative ring and let M, N, and P be A-modules. Prove that there exists a canonical isomorphism of A-modules $M \otimes (N \oplus P) \simeq (M \otimes N) \oplus (M \otimes P)$.
5. Let A be a commutative ring and let M be an A-module. Show that the following are equivalent:
   (i) M is a projective A-module.
   (ii) $\mathrm{Hom}_A(M, \_)$ is an exact functor.
   (iii) $\mathrm{Ext}^1_A(M, N) = 0$ for all A-modules N.