BU Algebra Qualifying Exam, Spring 2005
May 5, 2002, SR 243, 2:00 - 5:00 pm

(A) Write an essay on the Fundamental Theorem of Galois Theory. Include a complete statement, definition of terms, an outline of the proof (without details), and applications. Please use complete sentences.

CLEARLY STATE WHICH PROBLEMS YOU ARE SUBMITTING IN PARTS (B) THROUGH (E).

(B) Accurately state, without proofs, 5 of the following 8 theorems and give one non-trival, illustrative example of each (one example is sufficient even if there are several theorems stated).

B.1 The three Sylow theorems
B.2 The Jordan Hölder theorem for finite groups
B.3 The Krull-Schmidt-Azumaya Theorem for left R-modules
B.4 The four isomorphism theorems for left R-modules
B.5 The stacked basis theorem
B.6 Maschke's theorem
B.7 The Wedderburn-Artin theorem for left semi-simple rings
B.8 The fundamental theorem for finitely generated modules over principal ideal domains

(C) Give 5 of the following 8 definitions.

C.1 Solvable group
C.2 Jacobson radical of a ring
C.3 p-Sylow subgroup
C.4 Rational canonical form for a matrix
C.5 Separable field extension
C.6 Injective left R-module
C.7 Integral closure of an integral domain
C.8 The tensor product of a right R-module A and a left R-module B.

(D) Prove 5 of the following 8 statements. Any theorems you use in the proof should be clearly identified, either by name or statement.

D.1 A finite p-group has a non-trivial center.

D.2 Two $n \times n$ matrices A and B over a field k are similar if and only $xI - A$ and $xI - B$ are equivalent $k[x]$-matrices.

D.3 A left R-module M is an projective module if and only if $Hom_R(\overset{M}{\cancel{P}}, -)$ is an exact functor.

D.4 A left R-module M is the internal direct sum of two non-zero submodules A and B if and only if there are two non-trivial idempotents e and f in $End_R(M)$ with $1 = e + f$ and $ef = fe = 0$.

D.5 If k is a field, $f(x) \in k[x]$ has degree n, and E is the splitting field of $f(x)$, then $Gal(E/k)$ is isomorphic to a subgroup of S_n.

D.6 If V is a vector space and W is a subspace of V, then there is a subspace W' of V with $V = W \oplus W'$.

D.7 If M and N are two $n \times n$ matrices over a field k, then $\det(MN) = \det(M)\det(N)$.

D.8 If N is a submodule of a left R-module M, then M is a left artinian module if and only if both N and M/N are left artinian modules.

(E) Work 5 of the following 8 problems. Show your work and explain your solution

E.1 Find the integral closure of $\mathbb{Z}$ in the field $\mathbb{Q}(i)$.

E.2 Find the Galois group of $x^3 - 2 \in \mathbb{Q}[x]$.

E.3 Show that a group with 80 elements is not simple.

E.4 Find, up to isomorphism, all groups with 49 elements.

E.5 Let $F = \mathbb{Z}^3$ and N the subgroup of F generated by $\{(1,6,6),(2,2,4),(8,2,4)\}$. Find cyclic groups C_i with $F/N = C_1 \oplus \ldots \oplus C_n$.

E.6 Find the elementary divisors, invariant factors, minimal polynomial, characteristic polynomial, rational canonical form, and Jordan canonical form of the following $\mathbb{Q}$-matrix:

$$A = \begin{pmatrix} 0 & 0 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

E.7 Suppose that $f(x)$ is a monic irreducible polynomial in $\mathbb{Q}[x]$ with a root $a + bi \in \mathbb{Q}(i)$. Find the coefficients of $f(x)$ in terms of a and b.

E.8 Let kG be the group ring with $k = \mathbb{Q}$ and $G = \mathbb{Z}_9$. Write kG as a product of fields.