BAYLOR DEPARTMENT OF MATHEMATICS
ALGEBRA QUALIFYING EXAMINATION 2012

Provide detailed responses to the following questions. Clearly identify any theorems from the course text that you use in your responses. Even if you do not know how to complete a problem, indicate partial progress you have toward a solution - this progress may be a source of partial credit. Good luck!

1) Let M and N be normal subgroups of G such that $G = MN$. Prove that $G/(M \cap N) \cong (G/M) \times (G/N)$.

2) Let G be a group with $|G| = pqr$ where p, q, r are primes with $p < q < r$. Prove that G has a nontrivial normal subgroup.

3) Solve the simultaneous system of congruences

$$x \equiv 1 \bmod 8 \qquad x \equiv 2 \bmod 25 \qquad x \equiv 3 \bmod 81 .$$

4) Show that $p(x) = x^3 + 9x + 6$ is irreducible in $\mathbb{Q}[x]$. Let θ be a root of $p(x)$. Find the inverse of $1 + \theta$ in $\mathbb{Q}(\theta)$.

5) Let K be a field and V a vector space over K of dimension n. Let $A \in End_K(V)$. Show the following are equivalent:

a) the minimal polynomial of A is the same as the characteristic polynomial of A.

b) there exists a vector $v \in V$ such that $v, Av, \ldots, A^{n-1}v$ is a basis of V.

6) Let K be the splitting field of the polynomial $x^4 - 2$ over $\mathbb{Q}$ and let G be the Galois group of K over $\mathbb{Q}$.

a) Describe G.

b) Describe all subfields of K containing $\mathbb{Q}$.