The Method of Equal Proportions: Apportioning Seats of the United States House of Representatives

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The method currently used to apportion seats in the United States House of Representatives might one day end in a tie when it is applied to census data. Prior planning regarding such an event could help avoid an unpleasant political confrontation. In this article, a description of why a tie can happen is presented, and three simple suggestions for breaking a tie are offered.

The method currently used is called the Method of Equal Proportions. Following the census of 1940, Congress adopted this method to apportion seats in the House of Representatives. In 1928 E. V. Huntington described a procedure for implementing the method [2], and a version of this procedure is still used by the Census Bureau [3].

E. V. Huntington

Let k denote the number of states and assume there is an ordered arrangement of the states where the i-th state has population P_i and $P_1 \geq ... \geq P_k$. If H is the size of the House of Representatives, it is necessary to require that $H \geq k$ in order to satisfy the constitutional mandate that each state must be apportioned at least one seat. Define $d(a) = \sqrt{a(a+1)}$ for each nonnegative integer a. Then $a \leq d(a) < a+1$ for each a. The array

$\frac{P_1}{d(1)}$	$\frac{P_1}{d(2)}$	$\frac{P_1}{d(3)}$	$\cdots$
$\frac{P_2}{d(1)}$	$\frac{P_2}{d(2)}$	$\frac{P_2}{d(3)}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{P_k}{d(1)}$	$\frac{P_k}{d(2)}$	$\frac{P_k}{d(3)}$	$\cdots$

is called the Huntington array associated with the population vector $(P_1, \cdots, P_k)$. The values in the cells of this array are the priority values associated with the population vector $(P_1, \cdots, P_k)$. Since d is a strictly increasing function, each row of the Huntington array is strictly decreasing. Moreover, each column of the Huntington array is monotonically decreasing because $P_1 \geq ... \geq P_k$. The values in the cells of the Huntington array can be arranged as a monotonically decreasing sequence $V_1, V_2, V_3, \ldots$ where a value is listed as many times as it appears in the cells of the array. The sequence $V_1, V_2, V_3, \ldots$ is called the priority-value sequence associated with the population vector $(P_1, \cdots, P_k)$.

On web site [3] is a table of four columns with headers: house seat, state, state seat, and priority value. The priority-value column contains the priority-value sequence defined above. The column headed by "house seat" contains the integers 51, 52, $\cdots$, 440. The state with the largest priority value gets the 51st house seat, the state with the second largest priority value gets the 52nd house seat, etc... In this way, the Census Bureau associates a state with each house seat numbered 51 through 440. The Bureau goes a little beyond the current house size of 435 because Congress might decide to enlarge the size of the House of Representatives. This association of a state with each house seat is uniquely defined only in the case that the finite priority-value sequence $V_1, V_2, V_3, \ldots, V_{H-k}, V_{H-k+1}$ is strictly decreasing.

To illustrate Huntington's priority-value working rule in the case where $V_1, V_2, V_3, \ldots, V_{H-k}, V_{H-k+1}$ is strictly decreasing, consider a country with five states A, B, C, D, and E and with populations 5000, 4500, 3200, 2000, and 1500, respectively. Initially each state is allotted one seat, so the apportionment vector corresponding to house size 5 is (1,1,1,1,1). The first four terms, $V_1, \ldots V_4$ of the priority-value sequence are used to find the apportionment vector corresponding to each house size 6, $\cdots$,9. The terms $V_1, \ldots V_5$ are $\frac{5000}{\sqrt{2}}, \frac{4500}{\sqrt{2}}, \frac{3200}{\sqrt{2}}, \frac{5000}{\sqrt{6}}, \frac{4500}{\sqrt{6}}$ respectively. Since state A corresponds to the first term of the priority-value sequence, state A is allotted the sixth seat and the apportionment vector corresponding to house size six is (2,1,1,1,1). State B corresponds to the second term of the priority-value sequence and is allotted the seventh seat. The apportionment vector corresponding to house size seven is (2,2,1,1,1). At this stage, $\frac{3200}{\sqrt{2}}$ is the largest unused priority value and it corresponds to state C. Consequently, state C is allotted the eighth seat and (2,2,2,1,1) is the apportionment vector that corresponds to house size eight. Since $\frac{5000}{\sqrt{6}}$ is the next largest priority value, state A is allotted the ninth seat. So (3,2,2,1,1) is the apportionment vector that corresponds to house size nine.

There is a shortcut whenever $V_{H-k} \neq V_{H-k+1}$. If C is a positive real number, then $\frac{P_1}{C}, \cdots, \frac{P_k}{C}$ are called the quotas associated with C. If a is a nonnegative integer and $a \leq \frac{P_i}{C} \leq a+1$ then $\frac{P_i}{C}$ rounds to $a+1$ when $\frac{P_i}{C} > d(a)$ and rounds to a when $\frac{P_i}{C} < d(a)$. The goal is to find a common divisor C so that the sum of the values to which $\frac{P_1}{C}, \cdots, \frac{P_k}{C}$ round is the house size H. Fortunately, any value strictly between V_{H-k} and V_{H-k+1} gives a common divisor where the sum of the rounded values is H. Moreover if $\frac{P_i}{C}$ rounds to a_i, then $(a_1, \cdots, a_k)$ is the unique apportionment vector associated with house size H. This procedure of finding a common divisor and rounding is called the traditional rule of apportionment. The traditional rule can be used with $C = 820$ to show (6,6,4,2,2) is the apportionment vector corresponding to the population vector (5000,4500,3200,2000,1500) with house size 20 because $V_{15} = \frac{4500}{\sqrt{30}} > 820 > \frac{2000}{\sqrt{6}} = V_{16}$.

The priority-value working rule associates apportionment vectors with each house size greater than or equal k. The house size $k+j$ corresponds to the term V_j of the priority-value sequence. The apportionment vectors associated with house size $k+j$ depend on properties of the plateau (a subsequence with constant value where no longer constant subsequence has that value) of the priority-value sequence in which V_j sits and on the unique apportionment vector that corresponds to the house size just prior to the plateau that contains V_j.

Assume V_j is the first term of a plateau of length s of the priority-value sequence and that there is a unique apportionment vector associated with house size $k+j-1$. If $s \geq l$, one gets the $\binom{s}{l}$ apportionment vectors for house size $(k+j-1)+l$ by adding one to exactly l of the coordinates, corresponding to the requisite s states, of the unique apportionment vector for house size $k+j-1$. Observe that this rule gives $\binom{s}{s}$ apportionment vectors for house size $(k+j-1)+s$. Consequently if there is a unique apportionment vector corresponding to the house size just prior to the beginning of a plateau of the priority-value sequence, there is a unique apportionment vector corresponding to the house size at the end of that plateau.

To illustrate the priority-value working rule when $V_{H-k} = V_{H-k+1}$ apportion 60 seats to three states with populations 3500000, 600000, and 100000, respectively. Some terms of the priority-value sequence are $V_{55} = \frac{3500000}{\sqrt{48*49}} \approx 72168.78$ and $V_{56} = V_{57} = V_{58} = \frac{3500000}{\sqrt{49*50}} = \frac{600000}{\sqrt{8*9}} = \frac{100000}{\sqrt{1*2}} \approx 70710.68$.

Since $V_{58-3} > V_{58-3+1}$, the traditional rule can be used to show (49,8,1) is the apportionment vector for house size 58. Since $V_{56} = V_{57} = V_{58}$, the three states are equally entitled to the 59th house seat. So the apportionment vectors for house size 59 are (50,8,1), (49,9,1), and (49,8,2). The three apportionment vectors for house size 60 are (50,9,1), (50,8,2), and (49,9,2).

It may be disconcerting to obtain multiple apportionment vectors for certain house sizes. However, this outcome may be more palatable when associated with Huntington's pairwise comparisons of states' representations. See [2] and [1, pp. 100-104] for details about Huntington's pairwise comparisons. Let a_i denote the number of seats tentatively apportioned to state i and let P_i denote the population of state i. As presented in [1, p. 102] and when using the Method of Equal Proportions, the amount of inequality between states i and j is $\frac{a_i P_j}{a_j P_i} - 1$ whenever $\frac{a_i}{P_i} \geq \frac{a_j}{P_j}$. If state i is over-represented in comparison to state j, that is if $\frac{a_i}{P_i} \geq \frac{a_j}{P_j}$, then a transfer of a seat from state i to state j should be made if the amount of inequality is diminished. Huntington proves [2] that by making successive transfers that diminish the amount of inequality, " it is always possible to arrive at a final apportionment which cannot be 'improved' by any further transfer between two states." The apportionment vectors given by the priority-value working rule and associated with house size 60 and the population vector (3500000,600000,100000) are: (50,9,1), (50,8,2), and (49,9,2). These are precisely the apportionment vectors for house size 60 which do not admit a transfer between two states that diminishes the amount of inequality.

In case of the improbable event that a tie were to occur, in other words were $V_{H-k} = V_{H-k+1}$, what would be a good way to break the tie? Three suggestions are offered:

(1) If $V_{H-k} = V_{H-k+1}$, then temporarily reduce the house size to the house size just prior to the plateau that contains V_{H-k}. There is a unique apportionment vector corresponding to this adjusted house size.

(2) If $V_{H-k} = V_{H-k+1}$, then V_{H-k} belongs to a plateau of the priority-value sequence that has length $s \geq 2$. When V_{H-k} is the l -th term of this plateau, randomly select l of the requisite s coordinates of the unique apportionment vector that corresponds to the house size just prior to the plateau that contains V_{H-k} and add exactly one to each of these l coordinates. This yields an apportionment vector for house size H.

(3) If $V_{H-k} = V_{H-k+1}$ then temporarily increase the house size to the house size corresponding to the end of the plateau that contains V_{H-k}. There is a unique apportionment vector corresponding to this adjusted house size.

Cursory observations indicate that the likelihood of a tie (namely of $V_{H-k} = V_{H-k+1}$) when applying the Method of Equal Proportions to the decennial census is very small. However, no credible measure of this likelihood has been found.

[1] M.L. Balinski and H.P. Young, *Fair Representation*, 2nd edition, Brookings Institution, Washington D.C., 2001

[2] E.V. Huntington, The Apportionment of Representatives in Congress, *Transactions of the American Mathematical Society*, 30, no. 1:85-110, 1928

[3] www.census.gov/population/censusdata/apportionment/00pvalues.txt