Applied Mathematics I Name:____________________________
Ph.D. Qualifying Exam

Choose FOUR of the following problems.

1. (a) Is $\{(1,2,0,0,\dots),(0,0,1,2,0,0,\dots),(0,0,0,0,1,2,0,0,\dots),\dots\}$ a total orthogonal set in ℓ^2? Justify your answer.

   (b) Is $\{\sin(nx)\}$, $n=1,2,\dots$ a total orthogonal set in $L^2[0,\pi]$? In $L^2[0,2\pi]$? Justify your answers.

2. (a) Let X be an inner product space. Show that if M is a finite dimensional subspace of X, then $X = M \oplus M^{\perp}$.

   (b) Let n be a fixed positive integer and $Y := \operatorname{span}\{e^{\pm ikt}\}_{k=-n}^{n}$. For $f \in L^2[-\pi,\pi]$, find the projection of f onto Y and find the distance from f to Y.

3. (a) Give an example of a (nontrivial!) bounded linear operator and compute its operator norm. Prove your assertion.

   (b) Give an example of an unbounded linear operator. Prove your assertion.

4. Consider the operator $T: L^2[0,1] \to L^2[0,1]$, $(Tx)(t) := \int_0^1 K(t,s)x(s)\,ds$ for $K \in C([0,1]\times[0,1])$ satisfying $K(t,s) = \overline{K(s,t)}$. Prove that T is a compact, self-adjoint linear operator. (Hint: Arzela-Ascoli)

5. Let H be the Hilbert space which is the completion of the set

$$C^1[0,1] := \{f \in C[0,1] : f'(t) \in C[0,1]\}$$

   with norm

$$\|f\| := \left(\int_0^1 |f'(t)|^2\,dt + \int_0^1 |f(t)|^2\,dt\right)^{1/2}.$$

   (a) Show that any $f \in H$ may be represented by a continuous function . That is, any sequence $\{f_n(x)\} \subset C^1[0,1]$ which is Cauchy in H, converges uniformly to some function $f \in C[0,1]$.

   (b) Let $\varphi(f) := f(0)$ for all $f \in H$. Show that $\varphi \in H^*$.

   (c) Find $g \in H$ such that $\langle f,g\rangle = \varphi(f)$ for any $f \in H$.

Applied Math II - Qualifier Portion
Do 5 of the 7 problems.

1. a) Assume that A is an n by n symmetric matrix with eigenvalues ordered $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Show that $\lambda_1 = \max_{||x||=1} x^* Ax$.

b) If A is 5 by 5 and symmetric and has eigenvalues -1, 2, 3, 4 and 6, what is $\max_{dimV=2} \min_{x \in V and ||x||=1} x^* V x$ and explain why.

2. a) Let D be a diagonal matrix. How can you compute e^D? Show the derivation.

b) Same for computing e^A with A a diagonalizable matrix.

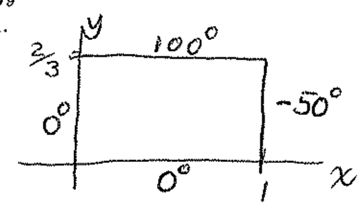
3. Let A be the 3 by 3 matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$

a) Compute the eigenvalues and eigenvectors of A.

b) Without multiplying any matrices together, explain how we know that $A^3 - 7A^2 + 10A = A(A - 2I)(A - 5I)$ equals the 3 by 3 zero matrix.

4. Suppose A is a positive definite symmetric matrix with eigenvalues ordered $\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n$. Discuss the convergence for power method versus a Krylov subspace method for computing the largest eigenvalue λ_1. (Hopefully draw some polynomials as part of this discussion). For what problems will the convergence be similar and when will it be quite different? Also if a shifted and inverted matrix is used, how will the convergence compare?

5. Consider the steady-state heat equation $u_{xx} + u_{yy} = 0$ on the domain shown with boundary temperatures specified as shown.



a) Using a finite difference approach with $h = 1/3$, find a system of linear equations. (It will be 2 by 2.)

b) Explain how you know ahead of time that Jacobi iteration will converge.

c) Take two iterations of Jacobi iteration with initial guess of the zero vector.

6. Consider this 1-D, elliptic boundary value problem:
$u'' = 2\delta(x - \frac{2}{3})$
$u(0) = 0$
$u(1) = 3$
Find the true solution and find the finite element solution with $h = \frac{1}{3}$.

7. a) For this 1-D wave equation with infinite domain:
$u_{tt} = c^2 u_{xx}$
$-\infty < x < \infty$
$u(x, 0) = g(x)$
$u_t(x, 0) = 0$
show that $.5(g(x - ct) + g(x + ct))$ is a solution.
b) Discuss how you could solve this problem numerically on a finite domain (such as $0 \leq x \leq 1$).