Topology Qualifying Exam
August 2010

Name: ________________________________

Directions: Do **exactly nine** problems. At least **four** must be chosen from Part 1 and at least **four** must be chosen from Part 2.

Part 1. Theorems from class:

(a) Every compact subspace of a Hausdorff space is closed.

(b) If X is path connected and $x_0, x_1 \in X$, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.

(c) Let $p : E \to B$ be a covering map and let $p(e_0) = b_0$. Any path $f : [0,1] \to B$ beginning at b_0 has a lifting to a path $\widetilde{f}$ in E beginning at e_0. [*Note:* You can skip uniqueness of the lift.]

(d) Let X be path connected. Attach a 2-cell B^2 to X by the map $\varphi : S^1 \to X$. In other words, let $Y = \left(X \coprod B^2\right) / \sim$ where $z \sim \varphi(z)$ for $z \in S^1$. Fix base points $x_0 \in X$ and $s_0 \in S^1 \subseteq B^2$ and let γ be a path from x_0 to $\varphi(s_0)$. By abuse of notation, write φ for the loop in X generated by φ (technically, the loop is given by $t \to \varphi(e^{2\pi i t})$). Let N be the normal subgroup generated by $\widehat{\gamma}[\varphi]$. Show the inclusion map $i : X \to Y$ induces an isomorphism

$$\pi_1(X, x_0)/N \cong \pi_1(Y, x_0).$$

(e) If $\{X_\alpha\}$ is the collection of path connected components of X, and if T_α is a fixed singular 0-simplex with image in X_α, then the homology classes $\{[T_a]\}$ form a basis for $H_0(X)$.

(f) (Main Lemma for Homotopy Invariance) There exists, for each space X and each non-negative integer p, a homomorphism

$$D_X : S_p(X) \to S_{p+1}(X \times I)$$

having the following properties:

1. If $T : \Delta_p \to X$ is a singular simplex, then

$$\partial D_X T + D_X \partial T = j_\#(T) - i_\#(T)$$

where $i(x) = (x, 0)$ and $j(x) = (x, 1)$.

2. D_X is natural; that is, if $f : X \to Y$ is continuous, then the following diagram commutes:

$$\begin{array}{ccc} S_p(X) & \to & S_{p+1}(X \times I) \\ \downarrow & & \downarrow \\ S_p(Y) & \to & S_{p+1}(Y \times I). \end{array}$$

(g) (Main Lemma for Excision) Suppose $\mathcal{A}$ is a collection of subsets of X whose interiors cover X. Then the inclusion map $\mathcal{S}^{\mathcal{A}}(X) \to \mathcal{S}(X)$ induces an isomorphism in homology.

Part 2. Homework-like problems:

(a) For $n \in \mathbb{N}$, let $x_n = (x_{n,\alpha})_{\alpha \in J}$ be a sequence in $X = \prod_{\alpha \in J} X_\alpha$ where each X_α is a topological space and X has the product topology.

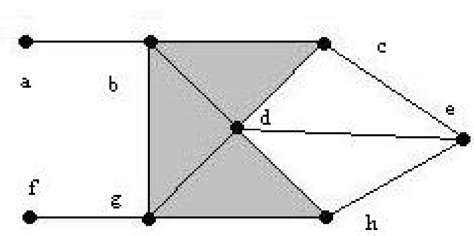
1. Show x_n converges to $x = (x_\alpha) \in X$ if and only if, for each α, the sequence $x_{n,\alpha} \in X_\alpha$ converges to x_α.
2. Show this statement can be false if X has the box topology.

(b) Let G be a topological group and H a subgroup of G. Show that the map $p : G \to G/H$ given by $p(g) = gH$ is an open map.

(c) Calculate:

1. $\pi_1(\mathbb{RP}^2)$
2. $\pi_1(T)$
3. $\pi_1(X)$ where X is the 2-cell B^2 equipped with the equivalence relation $z \sim z^3$ on $S^1 \subseteq B^2$. In other words, wrap the edge around on itself three times.

(d) Compute $H_1(K)$ and $H_2(K)$ of the simplicial complex K given by:



(e) If there is a retraction $r : K \to K_0$, show

$$H_p(K) \cong H_p(K, K_0) \oplus H_p(K_0).$$

(f) Let A be a closed subset of X and suppose A is a deformation retract of an open set in X. If X/A is the space obtained by collapsing A to a point, show that

$$H_p(X, A) \cong \widetilde{H}_p(X/A).$$

(g) Using CW complexes, calculate the homology groups of X where:

1. X is made up of three open cells–one in dimension 0, one in dimension 2, and one in dimension 4.
2. $X = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$