Real Analysis **Ph.D. Qualifying Exam** **May 31, 2010**

(1) (3 points each) Complete each of the following definitions.

(a) A *measure space...*

(b) A function f is *measurable...*

(c) A *positive measure...*

(d) The *integral of a nonnegative measurable function...*

(e) A *Hilbert space...*

(f) A *complete orthonormal system...*

(g) A real-valued function f is *absolutely continuous* on $[a, b]$...

(h) A family $\mathcal{A}$ of functions on X is said to *separate points...*

(i) The *total variation* of a real valued function f defined on the interval $[a, b]$...

(j) A linear operator A from a normed linear space X to a normed linear space Y is *bounded...*

(2) (3 points each) Give an example of each of the following, or state that no such example exists. You need not show any work.

(a) A $p \in [1, \infty]$ such that $L^p(\mu)$ fails to be a Banach space.

(b) A Hilbert space X and an orthonormal system in X that fails to be complete.

(c) A linear functional $F : L^2[0, 1] \to \mathbb{R}$ for which there fails to be a function g such that $F(f) = \int fg$ for all $f \in L^2$.

(d) A complete orthonormal system for $L^2[0, 2\pi]$.

(e) A linear functional on $L^1[0, 1]$ that is continuous at the identity on $[0, 1]$, but fails to be continuous at the Dirichlet function on $[0, 1]$.

(f) A sequence $\{f_n\}$ that converges to the zero function in $L^1(\mu)$, but fails to converge pointwise at any point.

(g) A sequence $\{f_n\}$ that converges pointwise to the zero function, but fails to converge to zero in the space $L^1(\mu)$.

(h) A sequence $\{f_n\}$ of measurable functions such that $\int \liminf f_n \, d\mu < \liminf \int f_n \, d\mu$.

(i) A decreasing sequence $\{A_n\}$ of measurable sets such that $\mu(\cap A_n) \neq \lim_{n\to\infty} \mu(A_n)$.

(j) An antiderivative that fails to be absolutely continuous.

(3) (10 points each) Prove two of the following.

(a) If $a < c < b$, then $T_a^b = T_a^c + T_c^b$.

(b) A real-valued function f has bounded variation on $[a, b]$ if and only if f is the difference of two monotone functions.

(c) If $A_1, A_2, A_3, ...$ are members of $\mathfrak{M}$, then $\mu(A_1 \cup A_2 \cup A_3 \cup ...) \leq \mu(A_1) + \mu(A_2) + \mu(A_3)....$

(d) Suppose $\{\varphi_n\}$ is an orthonormal system in a Hilbert space X. If $\{c_n\}$ is any sequence of real numbers with $\sum_{n=1}^{\infty} c_n^2 < \infty$, then there is $y \in X$ such that $(y, \varphi_n) = c_n$ for each $n \in \mathbb{N}$ and $y = \sum_{n=1}^{\infty} c_n \varphi_n$.

(4) (10 points each) Prove two of the following.

(a) Suppose $\{f_n\}$ is a sequence of mappings on a countable set D into a metric space Y such that, for each $x \in D$, the closure of $\{f_n(x) : n \in \mathbb{N}\}$ is compact. Then $\{f_n\}$ has a subsequence that converges for each $x \in D$.

(b) If $1 \leq p < \infty$, and if $\{f_n\}$ is a Cauchy sequence in $L^p(\mu)$, then $\{f_n\}$ has a subsequence that converges pointwise almost everywhere on X.

(c) (Do not appeal to the convergence theorems for this problem.) Suppose $\{f_n\}$ is a non-decreasing sequence of nonnegative measurable functions on a measure space $(X, \mathfrak{M}, \mu)$ that converges pointwise to a function f, φ is a measurable simple function supported by a set of finite measure such that $0 \leq \varphi \leq f$, and $0 < c < 1$. Then

(i) $A_1 \supset A_2 \supset A_3 \supset ...$ and $\cap A_n = \emptyset$ where $A_n = \{x \in X : f_n(x) < c\,\varphi(x)\}$ for each $n \in \mathbb{N}$, and

(ii) $\int_X c\,\varphi\, d\mu \leq \lim \int_X f_n\, d\mu$.

(5) (10 points each) Prove two of the following.

(a) Let f be a continuous periodic real-valued function on $\mathbb{R}$ with period 2π; that is, $f(x + 2\pi) = f(x)$ for all x. Show that, given $\epsilon > 0$, there is a finite Fourier series φ given by $\varphi(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)$ such that $|\varphi(x) - f(x)| < \epsilon$ for all x. [Recall that periodic functions are really just functions on the unit circle.]

(b) If $\{f_n\}$ is a sequence of functions in L^p $(1 \leq p < \infty)$ that is dominated by a function g in L^p and converges pointwise to f, then $f \in L^p$ and $\{f_n\}$ converges to f in L^p.

(c) Suppose $\{E_n\}$ is a disjoint sequence of measurable sets, and, for each n, suppose f_n is a function in L^p $(1 \leq p < \infty)$ that vanishes outside of E_n. Define $f = \sum f_n$. Then $f \in L^p$ if and only if $\sum ||f_n||^p < \infty$.