Algebra Qualifying Examination
May 24, 2010

Dr. Dugas Name________________________________

(A) State the definition:

(A1) The ideal I of the ring R is maximal, if...

(A2) Let E be a field extension of the field F. The polynomial $f(x) \in F[x]$ splits over E, if...

(A3) The field extension E of the field F is normal, if...

(A4) Let F be a field of characteristic $p > 0$. The Frobenius map $\varphi : F \to F$ is...

(A5) The (finite or infinite) group G is a p-group, if...

(A6) The subgroup P of the finite group G is a p-Sylow subgroup of G, if...

(A7) The field extension $K \subseteq F$ is Galois, if...

(A8) The left R-module M is a free module, if...

(A9) The left R-module M is semisimple, if...

(A10) $R = \bigoplus_{i=1}^{n} R_i$ is a ring direct product, if...

(A11) The ring R is simple, if R is semisimple and ...

(A12) The left R-module is indecomposable, if ...

(A13) If R is a ring, then the Jacobson radical of R is $J(R) = ...$

(A14) The ideal I of the ring R is nilpotent, if...

(A15) The ring R is a UFD, if...

(A16) Let E be a field and $a \in E^n$. Then the discriminant $D(a)$ is....

(A17) The ring R is a Dedekind domain, if...

(A18) Let I be an ideal of the commutative ring R. Then $\sqrt{I}$ is...

(A19) Let K be a field and I an ideal of the ring $K[x_1, ..., x_n]$. Then the variety $V(I) = ...$

(A20) In terms of maps, define: T is the tensor product of the two modules A, B, if...

(A21) Let R be an integral domain and K its field of fractions. Then I is a fractional ideal, if....

(B) State the indicated results:

(B1) Let J be a maximal ideal of the commutative ring R. Then R/J is a ....

(B2) Let E be a field extension of F and $g(x) \in F[x]$ irreducible over F. Let $\alpha, \beta \in E$ be roots of $g(x)$. Then there exists a map $\sigma : F[\alpha] \to F[\beta]$ such that.....

(B3) If every non-constant polynomial in $C[x]$ has a root in C, then C has no proper .... extensions.

(B4) Over a field of characteristic zero, every polynomial is....

(B5) The Orbit-Stabilizer Theorem.

(B6) The Sylow Theorems.

(B7) The Fundamental Theorem of Galois Theory.

(B8) The Primitive Element Theorem.

(B9) The First Isomorphism Theorem for left modules.

(B10) The Correspondence Theorem for left modules.

(B11) Schur's Lemma:

(B12) Jacobson's Theorem: If M is a semisimple R-module, $\varphi \in End_R(M)$ and $m_1, m_{2,}...., m_k \in M$, then ...

(B13) If I, J are simple left ideals of R, then $I \cong J$ as R-modules or else ....

(B14) Wedderburn's Theorem (on semisimple rings):

(B15) Maschke's Theorem:

(B16) The Jacobson radical $J(R)$ of the ring R is the largest ideal of R consisting entirely of .... elements.

(B17) R is Artinian and $J(R) = \{0\}$ if and only if ....

(B18) If R is a UFD, then R is .... closed.

(B19) Given the $AKLB$ setup. Under what condition is B a free A-module?

(B20) State the result on the factorization of fractional ideals in Dedekind domains.

(B21) State Hilbert's Basis Theorem.

(B22) State Hilbert's Nullstellen Satz.

(B23) State Wedderburn's Theorem on semisimple rings.

(B24) Let R be an Artinian ring. Then R is .....if and only if $J(R) = \{0\}$.

(C) Prove the following results:

(C1) If M is a simple R-module, then $End_R(M)$ is a division ring.

(C2) Let M be a semisimple left R-module and $A = End_R(M)$. If $\varphi \in End_A(M)$ and $m \in M$, then there is some $r \in R$ with $\varphi(m) = rm$.

(C3) Maschke's Theorem.

(C4) If R is Artinian, then $J(R)$ is nilpotent.

(C5) "Transitivity" of integral extensions of rings.

(C6) Let $f(x) \in F[x]$ be a non-constant polynomial. Prove that there is a field extension E of F in which $f(x)$ has a root.

(C7) Let E be a field extension of the field F and let $A = \{a \in E : a \text{ is algebraic over } F\}$. Prove that A is a field.

(C8) The Orbit Stabilizer Theorem.

(C9) If G is a finite group and p a prime number dividing the order of G, then G has a p-Sylow subgroup.

(C10) The Second Isomorphism Theorem for Modules

(C11) The theorem about the existence of dual bases.

(C12) A finitely generated module over a Noetherian ring is Noetherian.

(C13) If K is a field and $a_1, a_2, ..., a_n \in K$, then the ideal I of $K[x_1, x_2, ..., x_n]$ generated by $\{x - a_1, x - a_2, ..., x - a_n\}$ is a maximal ideal.

(C14) Given modules A_R and ${}_RB$ and T_1 and T_2 tensor products of A and B, show that $T_1 \cong T_2$.

(C15) In (C14) you proved that tensor products are unique. Now prove their existence.

Please work the following problems:

(D1) Show that $Q[\sqrt{3}, i] = Q[i + \sqrt{3}] =: E$. Find $[E : Q]$ and the minimal polynomial of $\alpha = i + \sqrt{3}$ over Q.

(D2) Let G be a group of order 77. Show that G is cyclic.

(D3) Let p be a prime and G be a finite group of order p^2. Show that G is abelian.

(D4) Find a Smith Normal Form for the integer matrix $A = \begin{bmatrix} 2 & 4 & 4 \\ -6 & 6 & 12 \\ 10 & -4 & -6 \end{bmatrix}$.

(D5) Same notations as in (D6). Write $Z^3/Col(A)$ as a direct sum of cyclic modules.

(D6) Given $A \in Mat_{n\times n}(Q)$ with $SNF(A - xI) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g(x) \end{bmatrix}$ where $g(x) = (x^2+1)(x-2)^2(x-3)^2$. Find the Rational Normal Form of A. Does A have a Jordan normal form? Why?

(D7) Given a field K and $S = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} : a_{ij} \in K \right\}$.

Prove that $J(S) = \left\{ \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix} : a_{ij} \in K \right\}$. What is $S/J(S)$?

(D8) Let R be a ring. Show that $J(R/J(R)) = \{0\}$.

(D9) Let B be an integral extension of the ring A. Show A is a field if and only if B is a field.

(D10) Let z be a root of the irreducible (over Q) polynomial $x^3 - 3x + 1$. Compute the norm $N(1 + z)$ in the field $Q[z]$.

(D11) Let $F = Q[\sqrt{3}]$, a quadratic field extension of the field Q of rational numbers. Explain why $B = Z[\sqrt{3}]$ is the ring of algebraic integers of F. Explain why $7B$ is a prime ideal of B, but $13B$ is not. Find prime ideals I, J of B such that $7B = IJ$.

(D12) Let R be a commutative ring and $I, J \trianglelefteq R$. Show that $(R/I) \otimes_R (R/J) \cong R/(I + J)$.

(D13) Let $Z[x]$ be the ring of integer polynomials. Prove: $Q \otimes_Z Z[x] \cong Q[x]$.

(D14) Let J be an ideal of the polynomial ring $K[x_1, ..., x_n]$. Show that $\sqrt{J} \subseteq IV(J)$.