Welcome back Calculus II. I am William Burns, your Calc II master tutor for the fall semester of 2022. I am myself a Mathematics major and hope you are excited about this class as I am. I look forward to seeing you in Sid Rich for group tutoring.

Group tutoring for Calc II this semester happens every Tuesday 5:15–6:15 pm in Sid Rich 074. For purposes of contacting me, my email is William_Burns1@baylor.edu.

Keywords: Differential Equation, Particular Solution, General Solution, Initial Condition, Separation of Variables, Exponential Growth/Decay

Topic of the Week: Differential Equations

Highlight: 9.1 Introduction to Differential Equations

A differential equation is an equation relating a function to one or more of its derivatives. In algebra, we are often interested in solving for the value of a variable. In differential equations, we are often interested in solving for the value of a function or the form of a function. A function that makes a differential equation true is called a particular solution to the differential equation, and the form of all solutions to the differential equation is called the general solution.

Example: $\frac{dy}{dx} = y$.

One will perhaps recall that there is only one kind of function that is equal to its own derivative: the exponential function. Therefore, a particular solution to this differential equation is $y = e^x$, and the general solution is $y = ce^x$, where $c$ is a constant.

A differential equation may also contain the variable of which $y$ is a function. For example,

$$\frac{dy}{dx} = 1 - 6e^{2x}$$

To solve this equation, we multiply both sides by $dx$ and then integrate to get the following,
\[ y = x - 3e^{2x} + C \]

This is the general solution to the differential equation (Rogawski 499). A particular solution would choose a particular value for the constant C. Sometimes, one is given some extra bit of information called an initial condition. The initial condition narrows down the range of possible solutions to just one particular solution. Thus, any of a number of particular solutions in the form of the general solution might solve the differential equation, but only one particular solution will solve the differential equation and also satisfy the initial condition.

There is no easy algorithm for solving differential equations. Rather, there is a handbag of techniques for approaching differential equations, depending on how they can be classified. For example, some first-order (involving just the first derivative) differential equations can be manipulated into the form

\[ \frac{dy}{dx} = f(x)g(y) \]

Such differential equations are called separable, and they have an easy method for solving. This method is called separation of variables, because it relies on the ability to factor y's derivative into a function of just x and a function of just y. The way forward from here is to multiply both sides by \( \frac{dx}{g(y)} \) to get \( \frac{dy}{g(y)} = f(x) \, dx \). All that is left to do from here is to integrate both sides, each with respect to its respective variable: \( \int \frac{dx}{g(y)} = \int f(x) \, dx \). After performing this integration, one should be able to solve for y in terms of x, which is the solution to the differential equation.

For example, consider the following differential equation

\[ \frac{dy}{dx} = 6xy^2 \]

Notice that the derivative of y with respect to x is factorable into a function of just x (6x) and a function of just y (y^2). Applying the algorithm for separation of variables,

\[
\frac{dy}{y^2} = 6x \, dx \\
\int \frac{dy}{y^2} = \int 6x \, dx \\
\int \frac{1}{y} = \int 3x^2 + C \\
y = -\frac{1}{3x^2 + C}
\]

This is the general solution to the differential equation (Rogawski 500).

Most differential equations are not solvable using separation of variables. However, for Calc 2 purposes, this is the only special method covered in the textbook. This means that unless your professor says otherwise, separation of variables is the main method you need to know for this class.
Differential equations have applications to many fields, especially physics. One common physical phenomenon describable with differential equations is exponential growth and decay. A function is said to grow exponentially when it is proportional to its own rate of change, in intuitive words: the more a thing grows, the faster it grows. The same applies for exponential decay, or shrinkage. This idea of a function’s proportionality to its rate of change may be simply stated using differential equations:

$$\frac{dy}{dt} = ky$$

Solving this differential equation using separation of variables, one will obtain the following general solution:

$$y(t) = De^{kt}$$

The above formula is simply a more general case of the very first differential equation we looked at. This is because the only function that is equal to its own derivative is the exponential function $e^x$.

Consider a commonly used example of exponential growth. Let us say that a population of E. coli bacteria grows exponentially. If the initial population is 1000, and there are 1500 present after 1 hour, what function describes the population of E. coli bacteria after $t$ hours?

Well, since the growth of the E. coli bacteria is exponential, if $p(t)$ is the function describing the population after $t$ hours, we know that $p(t)$ must be in the form $De^{kt}$.

We plug in our information on the initial population: $1000 = p(0) = De^{k\cdot0}$

And our information about the population at 1 hour: $1500 = p(1) = De^{k\cdot1}$.

This yields 2 equations and two unknowns. We solve for the parameters $D$ and $k$. After a little algebra, $D = 1000$ and $k = ln(1.5) \approx 0.405$ (Rogawski 503). Thus, the function requested is as follows:

$$p(t) = 1000e^{0.405t}$$

After recognizing the form of the differential equation, these exponential growth word problems really do not involve any more calculus. After getting the above equation, it is just a matter of remembering one’s algebra to solve for special times, population sizes, and initial populations — whatever specifics the problem may demand.

Check Your Learning

1. Solve the initial value problem: $y \frac{dy}{dx} = xe^{-y^2}, \ y(0) = -2$ (Rogawski 506)

2. Solve the initial value problem: $t^2 \frac{dy}{dt} - t = 1 + y + ty, \ y(0) = 0$ (Rogawski 506)
Things you may Struggle With

1. *Separation of variables*- These problems really test your algebra and order of operations more than anything. Be sure to remember the rules for manipulating exponentials (adding, multiplying, dividing, etc.) as you will have to do so frequently when getting these differential equations into a solvable form.

2. Word problems- Word problems requiring differential equations can be tricky the first time you see them. Draw pictures if necessary. Focus on giving important values mathematically symbolic names. *Figure out what $\frac{dy}{dx}$ corresponds to*. The intuition needed for these problems is similar to the intuition needed for related rates problems back in Calc 1.

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Answers to Check Your Learning

1. $-\sqrt{\ln(x^2 + e^4)}$

2. $y = \frac{et}{e^{1/t}} - 1$

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References