Howdy. This resource focuses on material covered in the twelfth week of classes, namely topics from sections 5.1 – 5.2 of OpenStax’s Precalculus. Please refer to other resources if your class is at a different point. Don’t forget to come to Group Tutoring on Mondays at 5:15 in room 75 of the basement of Sid Rich!

Keywords: Angles, Unit Circle, Sine, and Cosine Functions

Topic of the Week:

All about Angles!

An angle is “the union of two rays having a common endpoint”. Each ray starts at the endpoint and extends in a straight line from it out to infinity. The endpoint is also known as the angle’s vertex. When we draw angles, it is the convention to draw them in standard position, in which the vertex lies at the origin of the coordinate plane and the initial side lies on the positive x-axis. Also, angles can be positive or negative. Positive angles are measured in the counterclockwise direction, and negative angles are measured in the clockwise direction. For example, the angle in Figure 1 is a positive angle because the arc with the arrow is drawn in the counterclockwise direction. Another important convention to know when discussing trigonometry is the four quadrants of the coordinate plane. The quadrants are numbered in the counterclockwise direction as shown in Figure 2.

![Figure 1. Quadrants of the Coordinate Plane](image1.png)

![Figure 2. An Angle in Standard Position](image2.png)
Highlight #1: Drawing and Measuring Angles

The measure of an angle is “the amount of rotation from the initial side to the terminal side” (marked with the arc with the arrow like in Figure 1). There are two units used to measure angles: degrees and radians. A degree is “1/360 of a circular rotation” and an angle measure in degrees is marked with the symbol °. A radian is “the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle” See Figure 3. This means that when a 1-radian angle is drawn in standard position, and the tips of the rays touch a circle centered around the origin, the arc connecting the initial and terminal sides of the angle is equal in length to the radius of the circle.

![Figure 3. Measuring Radians](image)

We could prove that one full rotation is 360° or 2𝜋 radians. Therefore,

\[ 180° = \pi \text{ radians} \]

We can use the above equality to convert between degrees and radians. Note that if an angle measure is not given a unit, it is implied that the units are radians.

To draw an angle in standard position, use the following steps:

1. Divide the given angle measure by 360° or 2𝜋 radians, depending on the given units.
2. Reduce the fraction.
3. Rewrite the reduced fraction so that you can visualize the portion of the circle that it represents.
4. Draw the initial side of the angle on the positive x-axis.
5. Rotate counterclockwise if the angle is positive and clockwise if the angle is negative.
6. Draw the terminal side so that the angle contains the fraction that you calculated.
Coterminal Angles

Because angles rotate in a circle, once we rotate counterclockwise past 360° or $2\pi$ radians or clockwise past 0° or 0 radians, the angle is equivalent to an angle between 0° (0 radians) and 360° (2$x\pi$ radians). These equivalent angles are called coterminal. **Coterminal angles are “two angles in standard position that have the same terminal side”**. For a particular angle that is not between 0° (0 radians) and 360° (2$x\pi$ radians), we often want to find the coterminal angle between 0° (0 radians) and 360° (2$x\pi$ radians) because this range is easy to work with. If we are given an angle greater than 360° (2$x\pi$ radians), we can find the coterminal angle between 0° (0 radians) and 360° (2$x\pi$ radians) by subtracting 360° (2$x\pi$ radians) from the given angle until the angle is less than 360° (2$x\pi$ radians). If we are given an angle less than 360° (2$x\pi$ radians), we can find the coterminal angle between 0° (0 radians) and 360° (2$x\pi$ radians) by adding 360° (2$x\pi$ radians) to the given angle until the angle is greater than 0° (0 radians).

Reference Angles

**An angle’s reference angle** is “the size of the smallest acute angle, $t'$, formed by the terminal side of the angle $t$ and the horizontal axis”. A reference angle drawn in standard position will always be in quadrant I of the coordinate plane. **The calculation of an angle’s reference angle varies based on the quadrant in which the original angle lies**. See Figure 4.
Application of Angles

Angles can be applied to many applications involving circles and rotational motion.

I. Arc Length

An arc is a portion of the outline of a circle. The formula for arc length $s$ is

$$ s = r\theta $$

where $r$ is the radius of the circle that the arc is part of, and $\theta$ is the measure in radians of the angle that forms the arc. See Figure 5.

II. Area of a Sector

A sector is “a region of a circle bounded by two radii and the intercepted arc, like a slice of pizza or pie. See figure 6. To find a sector’s area, we can multiply the whole circle’s area ($\pi r^2$) by the fraction of the circle that the sector is. This results in the formula:

$$ A = \frac{1}{2} \cdot \theta \cdot r^2 $$

Note that $\theta$ must be in radians for the equation to be valid.

III. Linear and Angular Speed

When an object moves in a circle, it is said to be in rotational motion. An object in rotational motion has linear speed $v$ or “speed along a straight path” like all objects in motion. However, it also has angular speed $\omega$, which only objects in rotation motion have. Table 1 gives the formulas for linear speed, angular speed, and the relationship between the two.

<table>
<thead>
<tr>
<th>Linear Speed</th>
<th>$v = \frac{s}{t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angular Speed</td>
<td>$\omega = \frac{\theta}{t}$</td>
</tr>
<tr>
<td>Relationship Between Linear and Angular Speed</td>
<td>$v = r\omega$</td>
</tr>
</tbody>
</table>

Note that $s$ is in general the distance travelled by the object, but for circular motion it is also the arc length. $t$ is the time elapsed, $\theta$ is the angle traversed, and $r$ is the radius of the circular path. Also,
note that \( \omega \) is the lowercase Greek letter omega and should be in radians per unit time for the third equation in Table 1 to hold.

Highlight #2: Unit Circle: Sine and Cosine Functions

The unit circle is the circle with radius 1 centered at the origin. Figure 7 shows the unit circle with frequently used angles and the x- and y-coordinates where the angles’ terminal sides intersect the circle. Additionally, it is annotated with the signs of the x and y values in each quadrant. It is probably the most important figure in all of trigonometry. I highly recommend memorizing it, at least the first quadrant, from which all the other quadrants can be found.

If we call one of these angles \( \theta \), the point \((x, y)\) at which the terminal ray of the angle intersects the unit circle is given by

\[
x = \cos \theta \quad \text{and} \quad y = \sin \theta
\]

\( f(\theta) = \cos \theta \) is the cosine function, and \( f(\theta) = \sin \theta \) is the sine function. Note that these functions’ domain is all real numbers, and their range is \(-1 \leq \theta \leq 1\).

An important identity relating sine and cosine is the Pythagorean Identity:

\[
\cos^2 \theta + \sin^2 \theta = 1.
\]

This identity comes from the equation for the unit circle, \( x^2 + y^2 = 1 \).
Things you might struggle with:

1. Remember to convert from radians to degrees or vice versa, depending on the problem and what it is asking!
2. You might have the same angle, but based on whether or not it is negative or positive, there will be a different number.
3. Try to notice a pattern in the unit circle and this will help you memorize it!

Example Problems

1. Convert degrees and radians.
   a. 180°
   b. -60°
   c. π
   d. -3π

2. What is the arc length?
   a. The circle has a radius of 5 and an angle of 60°
   b. This circle has a radius of 2 and an angle of -90°

3. What is the area of the sector?
   a. This circle has radius of 10 and an angle of 30°

4. What is the sin and cos of the angle (without using a calculator)?
   a. 30°
   b. -30°
   c. 3 * π / 2
Example Problem Answers

1. The conversion between the 2 involves pi and 180°!
   a. 180 degrees to radians
      \[180 \times \pi / 180 = \pi\]
   b. -60 degrees to radians
      \[-60 \times \pi / 180 = -\pi / 3\]
   c. \(\pi\) radians to degrees
      \[\pi \times 180 / \pi = 180°\]
   d. -3\(\pi\) to degrees
      \[-3\pi \times 180 / \pi = -520° \text{ or } 180° \text{ (coterminal!)}\]

2. Look above to find the equation and explanation for arc length.
   a. \(r = 5\), theta = 60 degrees
      First you have to change the angle to radians
      \[60 \text{ degrees} = \pi / 3\]
      Now just plug into the equation!
      \[s = r\theta\]
      \[s = 5 \times \pi / 3\]
      \[s = 5/3 \times \pi\]
   b. \(r = 2\), theta = -90 degrees
      First you have to change the angle to radians
      \[-90 \text{ degrees} = \pi / 2\]
      Now just plug into the equation!
      \[s = r\theta\]
      \[s = 2 \times \pi / 2\]
      \[s = \pi \text{ (note that this value is not negative, length is always positive!)}\]

3. Look above to find the equation and explanation for sector area.
   a. \(r = 10\), theta = 30 degrees
      First you have to change the angle to radians
      \[30 \text{ degrees} = \pi / 3\]
      Now just plug into the equation!
      \[A = \frac{1}{2} \times \theta \times r^2\]
      \[A = \frac{1}{2} \times \pi / 3 \times 10^2\]
      \[A = \pi / 6 \times 100\]
      \[A = 50\pi / 3\]
   b. 

4. Using the unit circle, look at the coordinates and relate x and y to sin and cos.
   a. \(\cos(30) = \sqrt{3} / 2\)
      \(\sin(30) = 1/2\)
   b. \(\cos(30) = \sqrt{3} / 2\)
      \(\sin(30) = -1/2\)
c. $3 \pi / 2$ is equivalent to 270 degrees!
\[ \cos(270) = 0 \]
\[ \sin(270) = -1 \]

**Reminders**

- Group Tutoring for Precalculus is on Monday evenings from 5:15 to 6:15 in room 75 of Sid Rich! This is a great time to come ask specific questions to work together through some problems.
- The Tutoring Center has great Precalculus videos on YouTube! You can find links for them at [https://www.baylor.edu/case/index.php?id=978622](https://www.baylor.edu/case/index.php?id=978622)
- If you want individual help, you can also schedule a FREE 30-minute appointment with one of our awesome Precalculus tutors at [https://www.baylor.edu/tutoring](https://www.baylor.edu/tutoring)

All diagrams, tables, and external information is property of Precalculus by J. Abramson et al. unless otherwise specified.

Available: [https://openstax.org/details/books/precalculus](https://openstax.org/details/books/precalculus)