Hey everyone! My name is Savanah Smith, and I am the Master Tutor for Calculus III this semester. I hope these resources can become a great studying and review tool for everyone who encounters them. I will also be having weekly Group Tutoring sessions where we will go over the topics and practice problems covered in these resources, so feel free to check that out as described below! Please don’t hesitate to reach out to me with any questions or comments or when in doubt go to the tutoring website!

This week classes should be covering the beginning of Chapter 14 so we will be working through the topics in sections 14.3 and 14.4.

**Key Words:** Partial Derivatives, Tangent Planes, Differentiability

**TOPIC OF THE WEEK**

**Partial Derivatives**

A function \( f(x, y) \) therefore has two partial derivatives: \( f_x \) and \( f_y \)

- Notation: \( \frac{\partial f}{\partial x} \), this symbol is essentially a “rounded d” and is sometimes called “del”

The most important key to determining the partial derivatives is to know which variable you are taking the derivative of and treating the other variable(s) as constants.

**Ex:** Find the partial derivatives of \( f(x, y) = 4x^3 y^2 + 3x + e^y \)

Step 1: Find the partial in terms of \( x \)

\[
 f_x(x, y) = \frac{\partial}{\partial x}(4x^3 y^2 + 3x + e^y)
\]
Step 2: Treat the variable \( y \) as a constant

\[
\frac{\partial}{\partial x} (4x^3y^2 + 3x + e^y)
\]

- Treat \( y^2 \) as a constant so only the derivative of \( 4x^3 \) is taken and \( y^2 \) is left alone. (pretend \( y^2 = 2 \))

\[
\frac{d}{dx} (4x^3 \cdot 2) = 12x^2 \cdot 2 = 12x^2 \cdot y^2
\]

- Since there is no \( y \) in this term, just take the derivative like usual

\[
\frac{d}{dx} (3x) = 3
\]

- Since there is no \( x \) in this term, the whole thing is treated as constant (pretend \( e^y = 7 \))

\[
f_x(x, y) = 12x^2y^2 + 3 + 0
\]

Step 3: Find the partial derivative in terms of \( y \)

\[
f_y(x, y) = \frac{\partial}{\partial y} (4x^3y^2 + 3x + e^y)
\]

Step 4: Treat the variable \( x \) as a constant, using the same thinking as Step 2

\[
f_y(x, y) = 4x^3 \cdot 2y + 0 + e^y
\]

\[
f_y(x, y) = 8x^3y + e^y
\]

*All derivative rules (chain, product, quotient) still apply in the same way if the variable not being solved for is treated as a constant!
**HIGHLIGHT #1: HIGHER ORDER PARTIAL DERIVATIVES**

These are similar to regular higher order derivatives, except you must pay attention to which variable you are deriving for, for each subsequent derivation.

There are two types of higher order partials: second-order and mixed partials

**Second order:** taking the derivative of the same variable twice
- Notation: $f_{xx}$ or $f_{yy}$

$$f_{xx} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right)$$

**Mixed partials:** taking the first derivative in terms of one variable then taking the second derivative in terms of a different variable
- Notation: $f_{xy}$ or $f_{yx}$
- Be careful about the order! If it is written $f_{yx}$ then it goes **left to right**, if written $\frac{\partial f}{\partial y \partial x}$ then it goes **right to left**. Both ask to take the derivative of $x$, then $y$.

$$f_{xy} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)$$

- **Clairaut’s Theorem** states that if $f_{xy}$ and $f_{yx}$ both exist and are continuous then:

$$f_{xy} = f_{yx}$$

And this is true for any higher order differential, so:

$$f_{xxxx} = f_{xxyy} = f_{xyxx} = f_{yyxx} = f_{yxyx}$$

Because of this principle, we have the freedom to choose the order which we differentiate so choose wisely to make the problem easiest!

**Ex.** Calculate partial derivative of $g_{zzxy}$ where $g(x, y, z) = \arcsin \left( \frac{y^2 z}{e^z} \right) + x^3 y^2 z$

If we choose to take the partial in terms of $x$ first, the ugly arcsin term immediately goes away since there is no $x$ term in it which significantly simplifies the rest of the problem!

*Do $g_{xzzy}$ instead!*

Useful videos to further explain partial derivatives:

https://youtu.be/SbfRDBmyAMI
https://mathinsight.org/partial_derivative_examples
https://www.youtube.com/watch?v=BUlleGfqAeo

HIGHLIGHT #2: TANGENT PLANES & DIFFERENTIABILITY

Tangent Plane: tangent lines are for single variable equations \( f(x) \), and tangent planes are for functions with two variables \( f(x, y) \)
- Need a point \( (P) \) on the plane and a normal vector to define the plane
- The normal vector is the cross product of the two tangent vectors \( (u \text{ and } v) \) at the point on the plane

How do we find the tangent vectors?
- We can find one tangent line in the \( x \) direction and one in the \( y \) direction and find their slopes using the partial derivatives

General equation for the tangent plane if \( f(x,y) \) is locally linear: where \( a \) and \( b \) are the \( x \) and \( y \) coordinates of the point \( P \)

\[
z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)
\]

Locally linear: A plane is locally linear if as we zoom in on the graph at point \( P \), the graph looks flatter
- Observe the pictures on the right. The top picture is locally linear while the bottom is not.

Differentiability: if \( f_x(x,y) \) and \( f_y(x,y) \) exist and are continuous, then \( f(x,y) \) is differentiable on \( D \).
- Also it is differentiable at \((a,b)\), if it is locally linear

General Problem-Solving Steps:
1. Find the partial derivatives and determine if they exist and are continuous for all \((x,y)\)
   a. If so, then they can be considered locally linear and differentiable
2. Fill in the general equation for a tangent plane

The following videos are great resources to use for further explanation of tangent planes!

https://www.youtube.com/watch?v=pxmW8_Cpd7U


CHECK YOUR LEARNING

1. Find the partial derivative in terms of $x$ of $f(x, y) = e^{-x} + \sin (x + 2y)$

2. Find the partial derivative in terms of $y$ of the equation in #1.

3. Determine $g_{xxy}$ for $g(x, y) = e^{xy} + 3y$

4. Determine $g_{yxx}$ for the equation in #3.

5. Find the tangent plane of the graph of $f(x, y) = xy^3 + x^2$ at (1,3)

THINGS YOU MAY STRUGGLE WITH

1. Partial derivatives are difficult at first because it’s hard to imagine a variable as a constant. This will get easier with lots of practice and if you take your time and really think about what you need to do for each term. If it helps to rewrite the variables you are treating as constants as actually values, do it!

2. Some of the terms for tangent planes may sound difficult but it’s nothing you haven’t done before. Look for where the partials may be undefined, if there is nothing, then it is locally linear and differentiable, and you can proceed as normal with the general equation.

That’s all for this week! I hope this was a helpful review of Chapter 14.3 and 14.4! Feel free to visit the Tutoring Center Website for more information at www.baylor.edu/tutoring.

Answers:

1. $f_x = -e^{-x} + \cos (x + 2y)$
2. $f_x = 2\cos (x + 2y)$
3. $g_{xxy} = xy^2e^{xy} + 2ye^{xy}$
4. $g_{yxx} = xy^2e^{xy} + 2ye^{xy}$
5. $z = 29x+27-82$