MTH 1322: Calculus II
Week 15 Tutoring Resources

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Welcome Calculus II tutors and students! This will be the last resource for this semester. It has been a pleasure creating these resources and I hope that you all were able to learn from them. Since this is the last resource we will review the core topics of Calculus II. Additionally, the examples worked in this resource will be different from the ones worked in the previous resources. For more help with these topics please schedule a 1-on-1 visit with me or another tutor. If you would like to view any of the previous resources please click **HERE**. **Please visit baylor.edu/tutoring to make an appointment and to reserve a spot for the Calculus II group tutoring session every Tuesday at 6:30pm.**

Overview[1]

KEYWORDS: End of Semester Review: Taylor Series / Trig-Substitution / Disk and Washer Method

1 New Topics

1.1 Taylor Series

In last week's resource we reviewed Power, Taylor and Maclaurin series. We briefly worked some examples so in this resource we will continue with some more examples. Recall that we define a Taylor series expansion of a function $f(x)$ to be:

$$T_n(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \tag{1}$$

where a is the center of the series. For power series we let a_n be open for interpretation, but for Taylor Polynomials we must have:

$$a_n = \frac{f^n(a)}{n!} \tag{2}$$

Notice the f^n is the nth derivative of the expanded function f. Note: Maclaurin Series are Taylor Series centered at a = 0. For Taylor series we are not worried so much with convergence and divergence but rather we seek to express functions in the form of an infinite series.

Let's work an example. Suppose we are asked to find the Maclaurin series expansion of the following function.

$$f(x) = \sin(2x) \tag{3}$$

[1]The information used to create this resource was taken from this source: [1]

To find the Taylor series expansion we need to first consider the power series form of $\sin x$. It is important to note that it is only recommended to have memorized e^x as well as $\cos x$ and $\sin x$ and perhaps a few other common ones to be safe. If we start with the power series expansion of $\sin(2x)$ we can see we need to follow these steps: we first start with examining the derivatives of $\sin(2x)$. Notice that:

$$f'(x) = 2\cos(2x), \quad f''(x) = 2^2(-\sin(-2x)) \quad f'''(x) = 2^3 - \cos(2x) \quad f^{(4)}(x) = 2^4\sin(2x) \tag{4}$$

Notice that the pattern repeats after every 4th derivative since we are dealing with a trigonometric function. Since we were asked to find the Maclaurin series we should be evaluating these derivatives when $x = 0$. Therefore we have the following:

$$f'(x) = 2, \quad f''(x) = 2^2(0) = 0 \quad f'''(x) = 2^3 \quad f^{(4)}(x) = 2^4(0) = 0 \tag{5}$$

Thus we can a pattern develop. It is important to remember that we are indexing our derivatives from $n = 0$. For example note that $f^0(x) = f(x)$ so for $n = 1$ we have $f'(x)$. Therefore for even natural numbers we find the derivative at $x = 0$ is equal to 0 since every even derivative produces a $\sin(x)$ function. Furthermore we can also observe that every other odd derivative will produce a negative result. To account for our these results we can replace n with $2n + 1$. Now we can say that for our series:

$$a_n = \frac{f^n(0)}{n!} = \frac{2^{2n+1}(-1)^{2n+1}}{(2n+1)!} \tag{6}$$

For our next step we want to center our series at a = 0 since it is a Maclaurin Series.

$$\sum_{n=0}^{\infty} a_n(x)^n = \sum_{n=0}^{\infty} \frac{2^{2n+1}(-1)^{2n+1}}{(2n+1)!}(x)^{2n+1} \tag{7}$$

If we look closely we can see that this closely resembles the original power series expansion $\sin x$.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}x^{2n+1}}{(2n+1)!} \tag{8}$$

Another simpler way to do solve this problem would be to have the Taylor series for $\sin x$ memorized and from there plug in $2x$ in for x. Notice this yields the same result and with less steps.

$$\sum_{n=0}^{\infty} \frac{(-1)^{2n+1}(2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{2^{2n+1}(-1)^{2n+1}}{(2n+1)!}(x)^{2n+1} \tag{9}$$

Furthermore, if we were asked to find the radius of convergence for equation (7) we could do by using ratio test. Be careful when working with these problems as it may require you evaluate the first few derivatives by hand so that you can see the clear pattern.

If you need more help working with series please schedule a 1-on-1 appointment with myself or another tutor. To watch a video that works another example of Taylor series please click **HERE** [2]. If you have time I also recommend watching this slightly longer video that works multiple examples of Taylor series which you find by clicking **HERE** [3].

1.2 Solving Differential Equations

Since there are some sections that are covering differential equations for the first time we will work new examples. To view other examples with differential equations please view the older resources. Note that a simple equation of the form $\frac{dy}{dx} = f(x)$ is considered a differential equation that can be solved by simple integration.

For our first example, suppose we are asked to solve the initial value problem

$$y' = x(y^2 + 1) \quad y(0) = 0 \tag{10}$$

Our first step is to write y' as it is defined to be; $y' = \frac{dy}{dx}$. Now we have, $\frac{dy}{dx} = x(y^2+1)$. Next we separate variables and then integrate.

$$\frac{dy}{dx} = x(y^2+1) \implies \frac{dy}{y^2+1} = x\,dx \tag{11}$$

$$\frac{dy}{y^2+1} = x\,dx \Longrightarrow \int \frac{dy}{y^2+1} = \int x\,dx \tag{12}$$

$$\tan^{-1}(y) = \frac{1}{2}x^2 + C \tag{13}$$

Although we could solve for y and then use our initial condition to solve for C, it is actually easier to apply the initial condition before solving for C.

$$\tan^{-1}(0) = 0 + C \tag{14}$$

Since we have that $C = \tan^{-1}(0) = 0$, we can now find the particular solution to the differential equation by solving for y.

$$\tan^{-1}(y) = \frac{1}{2}x^2 \Longrightarrow y = \tan\frac{1}{2}x^2 \tag{15}$$

Most of the work done here is algebra so remember to take your time when solving these problems.

As aforementioned, be careful when solving these problems, since most of the work is algebra it can be easy to make a simple mistake. If you would like to watch a short video about solving differential equations please click **HERE** [4].

1.3 Trigonometric Substitution

In a previous resource we worked an example of all three cases of trig substitution but in this final review resource we will only solve one case. Let's solve the integral below as an example:

$$\int \frac{dx}{x^3\sqrt{x^2-4}} \tag{16}$$

Observe that because we have an integral of the form $x^2 - a^2$ we need to let $x = a\sec\theta$. In our case $a = 2$ so therefore we have $x = 2\sec\theta$ which means that $dx = 2\tan\theta\sec\theta$. Substituting this into our integral would look like the following:

$$\int \frac{dx}{x^3\sqrt{x^2-4}} = \int \frac{2\sec\theta\tan\theta\,d\theta}{\sec^3\theta\sqrt{4\sec^2\theta-4}} \tag{17}$$

Right away we are able to cancel out one of the $\sec\theta$ terms so that we are only left with $\tan\theta$ in the numerator. Our next step is to realize that we can make the trig substitution of $\sec^2\theta - 1 = \tan\theta$.

$$\int \frac{2\tan\theta\,d\theta}{\sec^2\theta\sqrt{4\sec^2\theta-4}} = \int \frac{2\tan\theta\,d\theta}{\sec^2\theta\sqrt{4(\sec^2\theta-1)}} = \int \frac{2\tan\theta\,d\theta}{\sec^2\theta\sqrt{4\tan^2\theta}} \tag{18}$$

Now we can simplify the denominator and cancel like terms.

$$\int \frac{2\tan\theta\,d\theta}{\sec^2\theta 2\tan\theta} = \int \frac{d\theta}{\sec^2\theta} = \int \cos^2\theta\,d\theta \tag{19}$$

From here there are two ways to proceed. If you have the reduction formula's memorized for integrals of $\cos^n\theta$ then you can simply write down the answer and substitute back in for theta. The other option is the know the double angle formula for $\cos^2\theta$. In this example we will use the double angle formula to integrate.

$$\int \cos^2\theta\,d\theta = \int \frac{1}{2} + \frac{\cos 2\theta}{2}\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \tag{20}$$

Our last step is solve for θ and $\sin 2\theta$. Notice that because $x = 2\sec\theta$ we can divide by 2 and then take the inverse secant to solve for theta. Lastly, since $\sec\theta = \frac{1}{\cos\theta}$ and $\cos = \frac{Adjacent}{Hypotenuse}$ we have that $\sec\theta = \frac{Hyp}{Adj}$. Therefore we can draw a right triangle with x as the hypotenuse and 2 as the adjacent side. From there we can use trigonometric identities to find a solution for $\sin 2\theta$. Putting this together we have the following as the final solution to our original integral.

$$\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta = \sec^{-1}(\frac{x}{2}) + \frac{1}{4}\frac{2}{x} \cdot \frac{\sqrt{x^2-4}}{x} \tag{21}$$

Students most often have trouble with converting their answer from terms of θ into terms of x. If possible I would recommend attempting to draw a triangle to explain how to convert from θ to x. To watch a video that covers Trig-Substitution click **HERE** [5].

1.4 Improper Integrals

In this final resource we will work another example of improper integrals. So suppose we are asked to determine if the following integral converges:

$$\int_3^6 \frac{x\,dx}{\sqrt{x-3}} \tag{22}$$

Notice that we can make a u substitution so that we can deal with the square root in the denominator. So let $u = x - 3$. Therefore we see that $du = dx$. However, notice that because we have x in the numerator we need to let $u + 3 = x$. Putting this together we have:

$$\int_3^6 \frac{u+3}{\sqrt{u}}\,du = \int_3^6 \frac{u}{\sqrt{u}}\,du + \int_3^6 \frac{3}{\sqrt{u}}\,du \tag{23}$$

Now observe that $\sqrt{u} = u^{1/2}$ so when we divide u by $u^{1/2}$ we have $u/u^{1/2} = u^{1/2} = \sqrt{u}$. This allows us to fully integrate and determine convergence.

$$\int_3^6 \sqrt{u}\,du + \int_3^6 \frac{3}{\sqrt{u}}\,du = \frac{2}{3}u^{3/2} + \frac{3}{2}u^{1/2} \tag{24}$$

Next we want to substitute $x - 3$ back in for u and evaluate the integral.

$$= \left(\frac{2}{3}(x-3)^{3/2} + \frac{3}{2}(x-3)^{1/2}\right)\Bigg|_3^6 = (\frac{2}{3}3^{3/2} + \frac{3}{2}3^{1/2}) - 0 = 13.8564063946 \tag{25}$$

Since we found the integral converges to a real definite number, we can say that our improper integral is convergent. Improper integrals require lots of practice to fully understand and apply ideas, that being said I recommend watching a short video to help refresh your memory. To watch the video please click **HERE** [6].

1.5 Volumes of Revolution (Disk & Washer Method)

In this section we review how to find the volume of a solid that was formed by rotating a region of the plane about an axis. Disk and washer method get their names from the type of object we are finding the volume of. If f is continuous and $f(x) \geq 0$ on [a,b], then the solid obtained by rotating the region under the graph about the x-axis has volume given by the general equation for disk method: [1]

$$V = \pi \int_a^b R^2 dx = \pi \int_a^b f(x)^2 dx \tag{26}$$

As an example, suppose we are asked to find the volume of $f(x) = \frac{1}{x^2}$ from $x = 1$ to $x = 4$ rotated about the x-axis. Looking at the image below we can see that we are asked to find the volume of the object created by rotating the highlighted region about the x-axis. By observation we see that our $f(x)$ satisfies the conditions to use the formula for disk method.

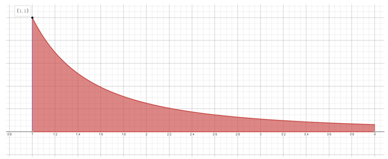

Figure 1: Region below $\frac{1}{x^2}$ [7]

Therefore we can say the volume of the object created is defined to be:

$$V = \pi \int_1^4 \left(\frac{1}{x^2}\right)^2 dx = \pi \int_1^4 \frac{1}{x^4}\, dx = \tag{27}$$

The main concern with these methods is the set up and not so much the calculation of the integral. Therefore I will refrain from showing the final step of integrating the polynomial. I recommend you work the rest of the problem out yourself to get more practice integrating. If you would like to check your work the answer to the above problem is: 1.03083508946

The next method we discuss, Washer method is merely a variation of Disk method. We use washer method when we are asked to find the volume of an object given by rotating the area between two curves. Suppose we have $f(x) \geq g(x) \geq 0$ where $f(x) - g(x)$ is the area between two curves; then we can define washer method to be:

$$V = \pi \int_a^b (R_{\text{outer}}^2 - R_{\text{inner}}^2) dx = \pi \int_a^b (f(x)^2 - g(x)^2) dx \tag{28}$$

Observe that if we are given functions in terms of y we could rewrite the equation to be as follows:

$$V = \pi \int_a^b (R_{\text{outer}}^2 - R_{\text{inner}}^2) dy = \pi \int_a^b (f(y)^2 - g(y)^2) dy \tag{29}$$

Let's work an example problem. Suppose we are asked to find the volume of a solid obtained by rotating the region between the graph of $f(x) = x^3$ and $f(x) = \sqrt[3]{x}$. Our first step should be to find where these two function intersect so that we know our bounds of integration. If we were to graph the two functions as we did below we would be able to find the intersection point. However, we find the intersection point without graphing. To do so we will set the functions equal to each other and solve for x:

$$x^3 = \sqrt[3]{x} \implies x^6 = x \implies x = 1 \tag{30}$$

Naturally finding the intersection for these two functions was rather trivial and could've done by inspection. However, this idea will come in handy for when things are not so trivial.

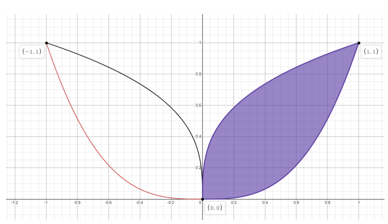

Figure 2: $y^3 - \sqrt[3]{y}$ rotated about y-axis [7]

Since we are rotating about the y-axis and using washer method, we will integrating with respect to y. Therefore we need to convert from terms of x into terms of y. Thus we have:

$$y = x^3 \longrightarrow x = \sqrt[3]{y} \tag{31}$$

$$y = \sqrt[3]{x} \longrightarrow x = y^3 \tag{32}$$

Now lets determine what the inner and outer radii will be. As a general rule, the radii for both washer and disk are always perpendicular to the axis of rotation. Therefore, we see our outer radius is $x = \sqrt[3]{y}$ and our inner radius is $x = y^3$; since we are rotating about y-axis. Thus we can find the volume by solving the following integral:

$$V = \pi \int_0^1 ((y^3)^2 - (\sqrt[3]{y})^2)dy \tag{33}$$

The exact solution can be found by solving the integral which will can simply be solved by hand or calculator so we will not solve for the exact solution.

Khan Academy provides great videos with some great illustration to better explain both Disk and Washer method. To watch the video series on Disk method click **HERE** [8]. To watch the series on Washer method click **HERE** [9].

References

[1] J. Rogawski, C. Adams, and R. Franzosa, *Calculus: Early Transcendentals*, 4th ed. New York: W. H. Freeman, Dec. 2018.

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