# MTH 1322: Calculus II
# Week 10 Tutoring Resources

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Welcome Calculus II tutors and students! In this week's resource we will be introducing sequences and then logistics equation. For more help with these topics please schedule a 1-on-1 visit with me or another tutor. **Please visit baylor.edu/tutoring to make an appointment and to reserve a spot for the Calculus II group tutoring session every Tuesday at 6:30pm.** If you would like to view any of the previous resources please click **HERE**.

**Overview**[1]



**KEYWORDS:** Logistics Equation / Sequences

## 1 New Topics

### 1.1 Sequences

As we continue moving forward we come across sequences. Sequences as we will soon see, is the foundation for infinite series and working with summing infinite series. In the upcoming topics relating to sequences, we will need to rely heavily on limits and convergence. Concepts we know from calculus 1 as well some topics we discussed while covering improper integrals will likely be used to help us deal with sequences. Let's begin by defining sequences, as the book defines it: A sequence $\{a_n\}$ is an ordered collection of numbers defined by a function $f$ on a set of sequential integers. The values $a_n = f(n)$ are called **terms** of the sequence, and $n$ is called the **index**. [1]. We typically only consider sequences where $n$ is a non-negative integer, i.e. $n = 0, 1, 2 \ldots$. Let's look at some examples of sequences:

$$a_n = \frac{n}{n+1}, \quad n \geq 0 \text{ with the first terms being: } 0, \frac{1}{2}. \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots \tag{1}$$

As you can see all we are doing is plugging in different values of $n$ into our function; we are not adding each term but rather keeping each term separate. Building off of this the next step is to determine convergence or divergence. To determine convergence we simply evaluate the sequence as $n$ approaches infinity. Therefore consider:

$$\lim_{n \longrightarrow \infty} \frac{n}{n+1} \tag{2}$$

If we use our knowledge of limits from calculus 1 we can see that the sequence converges to an indeterminate form and so we must use L'Hopital's Rule to evaluate this limit.

$$\lim_{n \longrightarrow \infty} \frac{n}{n+1} = \lim_{n \longrightarrow \infty} \frac{1}{1} = 1 \tag{3}$$

[1]The information used to create this resource was taken from this source: [1]

Therefore after L'Hopital's Rule we can see the limit is 1 as n approaches infinity. It is worth noting that our previous known properties of limits still apply, with an added property since we are now working with sequences. If we are given two convergent sequences, such that:

$$\lim_{n \longrightarrow \infty} a_n = L \quad \lim_{n \longrightarrow \infty} b_n = M \tag{4}$$

then it follows that:

$$\lim_{n \longrightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \longrightarrow \infty} a_n}{\lim_{n \longrightarrow \infty} b_n} = \frac{L}{M} \tag{5}$$

Consequently this bring us to the idea of a bounded sequence. We say that a sequence is bounded from above if there is a number $M$ such that the sequence, $a_n \leq M$ for all $n$, and as such we define $M$ as the upper bound.Similarly we say that a sequence is bounded from below if there is a number $m$ such that the sequence, $a_n \geq m$ for all n, and as such we define $m$ to be the lower bound. If a sequence is bounded then it is both bounded from above and below. A direct corollary to these statements is that if a sequence convergences then it is bounded. However, it is important to note that $M$ does not have to equal $L$.

For example, lets analyze the sequence $a_n = \frac{1}{n}$. Although some of you can make conclusions based on observation, it is always helpful to determine the first few terms of a sequence to better help us draw a conclusion. We see that the first few terms of $a_n$ for $n \geq 1$ would look like this:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \tag{6}$$

As you can see $a_n$ is decreasing. Furthermore, if we evaluate the limit as $n$ approaches infinity, we see that the limit converges to 1. Therefore the sequence is bounded from above by 1 and bounded below by 0. We also see that an upper bound for this sequence could be any number greater than 1.

Lastly, it is helpful if we can determine if a sequence is increasing or decreasing. We consider a sequence increasing if $a_n < a_{n+1}$ for all $n$. Similarly, we consider a sequence to be decreasing if $a_n > a_{n+1}$. Given we know either of these two about a sequence then we know one of the following to be true:

$$\text{if } a_n \text{ is increasing and } a_n \leq M, \text{ then } a_n \text{ converges and } \lim_{n \longrightarrow \infty} a_n \leq M \tag{7}$$

Or

$$\text{if } a_n \text{ is decreasing and } a_n \geq m, \text{ then } a_n \text{ converges and } \lim_{n \longrightarrow \infty} a_n \geq m \tag{8}$$

We call this theorem the monotone convergence theorem.

A common problem that can be asked might be for you to determine the formula of a sequence given the first few terms of a sequence. Let's discuss how to solve such problems. Consider the following first terms of a sequence $a_n$:

$$a_n = 1, -\frac{1}{8}, \frac{1}{27}, -\frac{1}{64} \ldots \tag{9}$$

Looking at this sequence we can see the first even terms of the sequence are negative and so since the sequence alternates between positive and negative we know the sequence will have the following in it: $(-1)^{n+1}$. Although it may not be extremely obvious we can see that if we took cubic roots of each term we would find the index as the answer. Therefore, we can describe the sequence as the following:

$$a_n = \frac{(-1)^{n+1}}{n^3} \tag{10}$$

In only a short time we will be discussing summing infinite series, therefore it is imperative that you fully understand the concepts we deal with in this chapter. I recommend watching this short video explaining sequences. To watch the video please click **HERE** [2]. As always, if you still find yourself confused please make an appoint with me or another tutor.

## 1.2 Logistics Equation

In last week's resource we discussed using differential equations to model populations of certain species. The model we introduced in our last resource models populations using exponential growth and decay. However, in the long run the model breaks down as it implies that there can be growth without limit. Looking at the model:

$$y = e^{kt}C \tag{11}$$

we see that for $k > 0$ the model depicts the population consistently growing at an exponential rate. We can see that this not possible as there are limiting factors to all species such as land and and food. Mathematically we can this is true by taking the limit as $t$ goes to infinity.

$$y = \lim_{x \longrightarrow \infty} Ce^{kt} = \infty \tag{12}$$

In other words we can see that as time progresses the population will also increase exponentially. Naturally this is not the case so we must find a different way to model such things. The logistic equation is merely an example of a model for a population. There are a plethora of different differential equations that can used to model populations of different species. Solving the general form of the logistics differential equation:

$$\frac{dy}{dt} = ky(1 - \frac{y}{A}) \tag{13}$$

will give us a general solution to model the growth of populations. Note that if we are given the max population, $A$, we can find the total population at any time $t$ using the general solution. We find that the general solution of the logistic equation is of the form:

$$y = \frac{A}{e^{-kt}\frac{1}{\beta} + 1} \tag{14}$$

If you would like to see a detailed derivation of the general solution please click **HERE**. Let's Work an example to help solidify this concept. Suppose we are given the following equation to model a population:

$$\frac{dy}{dt} = 3y(6 - y) \tag{15}$$

Before we can convert this differential equation into the general solution we must first realize that we must have $(1 - \frac{y}{A})$ in our equation. Therefore we must divide by 6 to reach our desired form.

$$\frac{dy}{dt} = \frac{1}{2}y(1 - \frac{y}{6}) \tag{16}$$

Now that we have a differential equation the desired form we can now apply the general solution. Note that because we have the desired differential equation form and we already derived a general solution that we are allowed to plug in variables into the general form of the Logistics Equation.

$$y(t) = \frac{6}{e^{-(1/2)t}\frac{1}{\beta} + 1} \tag{17}$$

Recall that $\beta$ is a constant of integration that appears and can be found when we apply an initial condition. Consider we have are asked to find the particular solution given the initial condition: $y(0) = 4$. For our next step we want to want plug in 0 for $t$ and plug in 4 for $y$ so that we can solve for $\beta$.

$$4 = \frac{6}{\frac{1}{\beta} + 1} \tag{18}$$

Next we want to multiply both sides by $(\frac{1}{\beta} + 1)$ and then divide by 4 to get

$$4(\frac{1}{\beta} + 1) = 6 \tag{19}$$

$$(\frac{1}{\beta} + 1) = \frac{3}{2} \tag{20}$$

Now if we subtract 1 from both sides we find that $\frac{1}{\beta} = \frac{1}{2}$ therefore $\beta = 2$. Thus the particular solution to our original problem is the following:

$$y(t) = \frac{6}{e^{-(1/2)t}\frac{1}{2} + 1} \tag{21}$$

If you find yourself lost by this explanation I recommend watching a short video. The following video will discuss deriving the general solution and also provide other explanation regarding the variables used. **HERE** [3].

## References


[1] J. Rogawski, C. Adams, and R. Franzosa, *Calculus: Early Transcendentals*, 4th ed. New York: W. H. Freeman, Dec. 2018.

[2] "What is a Sequence? Basic Sequence Info." [Online]. Available: https://www.youtube.com/watch?v=Kxh7yJC9Jr0

[3] "Logistic Differential Equation (general solution)." [Online]. Available: https://www.youtube.com/watch?v=TuQnl4RMnDk