MTH 1322: Calculus II
Week 8 Tutoring Resources

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Welcome Calculus II tutors and students! In this week's resource we will continue working with arc length and surface area as well as solving differential equations. For more help with these topics please schedule a 1-on-1 visit with me or another tutor. **Please visit baylor.edu/tutoring to make an appointment and to reserve a spot for the Calculus II group tutoring session every Tuesday at 6:30pm.** If you would like to view any of the previous resources please click **HERE**.

Overview[1]

KEYWORDS: Arc Length / Differential Equations

1 New Topics

1.1 Arc Length and Surface Area

Last week we reviewed the basics of arc length and surface area so this week we will work examples to help solidify these concepts. Suppose we are asked to find the arc length over the given interval:

$$f(x) = \ln \cos x \quad [0, \frac{\pi}{4}] \tag{1}$$

Recall that the formula for finding arc length is:

$$s = \int_a^b \sqrt{1 + f'(x)^2}\, dx \tag{2}$$

Therefore our first step is to differentiate our function. Using chain rule we have the following:

$$f'(x) = -\frac{\sin x}{\cos x} = -\tan x \tag{3}$$

Notice that since $f(x)$ is continuous on $[0, \frac{\pi}{4}]$ and $f(x) \geq 0$, we have that we can apply equation (2) to our equation.

$$s = \int_0^{\pi/4} \sqrt{1 + (-\tan(x))^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \tan^2(x)}\, dx \tag{4}$$

Now we can use the trigonometric identity to replace $1 + \tan^2 x$ with $\sec^2 x$. Thus we have

$$s = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx \tag{5}$$

[1]The information used to create this resource was taken from this source: [1]

Thus after integrating using the identity for the integral of $\sec x$ we have the following:

$$s = \int_0^{\pi/4} \sec x\, dx = \ln|\tan x + \sec x| \Big|_0^{\pi/4} = \ln|\tan\frac{\pi}{4} + \sec\frac{\pi}{4}| - \ln|\tan 0 + \sec 0| \tag{6}$$

Therefore our final solution is:

$$s = \int_0^{\pi/4} \sqrt{1 + (-\tan(x))^2}\, dx = \ln(1+\sqrt{2}) - \ln(1) = \ln(1+\sqrt{2}) \tag{7}$$

To watch a short video working a different example please click **HERE** [2].

1.2 Solving Differential Equations

In calculus 2 we only briefly touch on solving differential equations since there are multiple courses covering different forms. Differential equations are useful in describing exponential decay or half life problems, modeling a population of a species, as well as finding things such as how long it takes to empty a water tank. We will only discuss how to solve a differential equation used to model exponential growth and decay as well solve separable first order equations using separation of variables. It is worth noting that a simple equation of the form $\frac{dy}{dx} = f(x)$ is considered a differential equation that can be solved by simple integration [1].

For our first example, suppose we are asked to solve the initial value problem [1]:

$$y' = y^2 \sin x \quad y(\pi) = 2 \tag{8}$$

Our first step is to write y' as it is defined to be; $y' = \frac{dy}{dx}$. Now we have, $\frac{dy}{dx} = y^2 \sin x$. Next we separate variables and then integrate.

$$\frac{dy}{dx} = y^2 \sin x \implies \frac{dy}{y^2} = \sin x dx \tag{9}$$

$$\frac{dy}{y^2} = \sin x dx \implies \int \frac{dy}{y^2} = \int \sin x\, dx \tag{10}$$

$$-y^{-1} = -\cos x + C \tag{11}$$

Although we could solve for y and then use our initial condition to solve for C, it is actually easier to apply the initial condition before solving for C.

$$-(2)^{-1} = -\cos\pi + C \implies -\frac{1}{2} = -1 + C \implies \frac{1}{2} = C \tag{12}$$

Since we have that $C = \frac{1}{2}$, we can now find the particular solution to the differential equation by solving for y.

$$-y^{-1} = -\cos x + \frac{1}{2} \implies -1 = y(\cos x + \frac{1}{2}) \implies y = -\frac{1}{\cos x + \frac{1}{2}} \tag{13}$$

Most of the work done here is algebra so remember to take your time when solving these problems.

To model exponential growth or decay we will have a differential equation of the form

$$\frac{dy}{dt} = ky \tag{14}$$

We will still use separation of variables to solve this differential equation. Dividing both sides by y and multiplying both sides by dt we have $\frac{dy}{y} = kdt$. Integrating both sides we find that we have

$$\ln y = kt + C \tag{15}$$

To solve for y we need to raise both sides by e.

$$e^{\ln y} = e^{kt+C} \implies y = e^{kt+C} = e^{kt}e^{C} \tag{16}$$

Notice that e^C is just an arbitrary number which can be denoted by any variable but since the book uses D ($e^C = D$) so will we. Therefore the general solution a differential equation of the form: $\frac{dy}{dt} = ky$ is the following:

$$y(t) = De^{kt} \tag{17}$$

As aforementioned, be careful when solving these problems, since most of the work is algebra it can be easy to make a simple mistake. If you would like to watch a short video about solving differential equations please click **HERE** [3].

References

[1] J. Rogawski, C. Adams, and R. Franzosa, *Calculus: Early Transcendentals*, 4th ed. New York: W. H. Freeman, Dec. 2018.

[2] "Worked example: arc length | Applications of definite integrals | AP Calculus BC | Khan Academy." [Online]. Available: https://www.youtube.com/watch?v=OhISsmqv4_8

[3] "Separable First Order Differential Equations - Basic Introduction." [Online]. Available: https://www.youtube.com/watch?v=C7nuJcJriWM&t=75s