Calculus 1, Week 7

Hey Calculus tutors and students! This resource covers the sixth week of class. Specifically, in this resource I will cover how to use do Related Rates problems.

In addition to this resource and our one-on-one tutoring sessions, the Baylor Success Center also runs a weekly group tutoring session for Calculus 1. The group tutoring session occurs every Wednesday from 5pm-6pm. If you are interested in attending you can reserve a spot through our website:
https://www.baylor.edu/support_programs/index.php?id=40917

As with last year, if anyone has any suggestions for things to add to the resource please do not hesitate to contact me!

- Alex Williamson

Keywords: Related Rates.

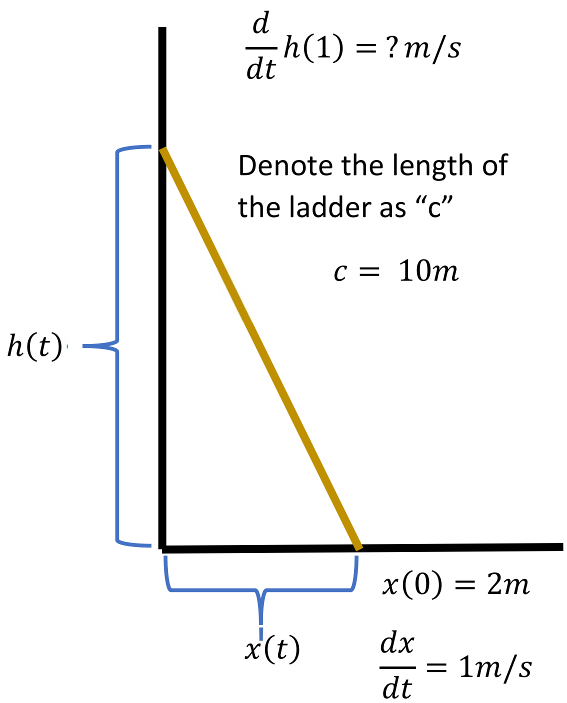
Key

- Yellow Highlighting: Definitions that you need to know.
- Green Highlighting: Explanation of how you actually go about doing the problems.
- Blue Highlighting: Helpful insight into why certain definitions work, how to think about problems, etc.

Concepts

- **Chapter 3.10**
  - Chapter 3.10 is the section of the book devoted to teaching the students Related Rates. This is generally one of the most difficult sections of the book for students to learn, and as a result the professor may devote extra time to going over it. Despite how difficult it is, at its core Related Rates problems are very simple. The easiest way to solve a Related Rates problem is by drawing a picture, determining exactly what you are solving for, then figuring out from your picture a formula that allows you to put what you want into terms of what you already know. This is made more difficult because you are often given the value of a mix of both constants and derivatives, so using implicit differentiation to get what you want into terms of what you have is key.
  - Video Resources
    - https://www.khanacademy.org/math/ap-calculus-ab/ab-diff-contextual-applications-new/ab-4-4/v/rates-of-change-between-radius-and-area-of-circle
  - Worked Example

- A ladder is against a building and slowly sliding down. Let h(t) be the height of the ladder at time t and let x(t) be the distance between the base of the building and the ladder at time t. The ladder is 10 meters long, the base of the ladder is 2 meters away from the house at t = 0, and the speed at which the base of the ladder is moving away from the building is 1 m/s. Find the speed at which the height of the ladder is changing at t = 1.



- The first step of a Related Rates problem is to write what we are looking for in mathematical terms. We are asked to find "the speed at which the height of the ladder is changing at t=1." The height of the ladder is h(t), so its speed will be $\frac{d}{dt}h(t)$. It's speed at t=1 would be written as $\frac{d}{dt}h(1)$
- The second step of a Related Rates problem is to draw a picture and label the diagram with all the information we were given. I also like to write out what we are solving for at the very top so I don't forget. Such a diagram is pictured to the left.
- The third step of a Related Rates problem is to identify an equation that we can use to relate all of the information that we have to the thing we are trying to solve for. Most of the time this equation will be a geometric equation of some kind. In this case, we will be using the Pythagorean Theorem ($h(t)^2 + x(t)^2 = c^2$). By taking the derivative of both sides in terms of t we get the following:

$$2h(t)\frac{dh(t)}{dt} + 2x(t)\frac{dx(t)}{dt} = 2c\frac{dc}{dt}$$

Since C is a constant we are left with just:

$$2h(t)\frac{dh(t)}{dt} + 2x(t)\frac{dx(t)}{dt} = 0$$

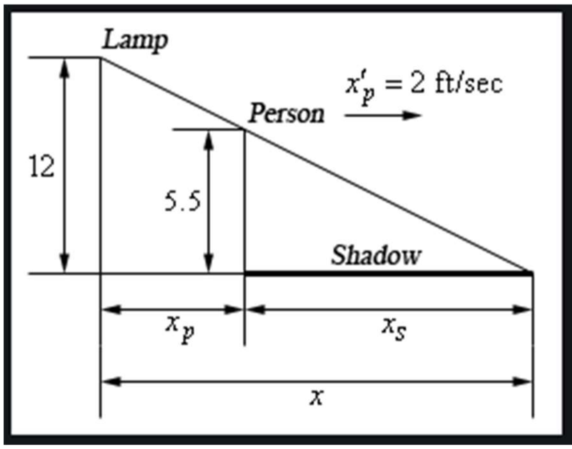
- However, we are quickly faced with a minor problem – while we are given the distance between the base of the ladder and the house at t = 0, $x(0)$, we really care about what the situation looks like at t = 1. Thankfully enough, we are also given the rate of change of x, $\frac{dx}{dt}$. Since the ladder is moving one meter away from the house every second, we know that at $x(1)$ the ladder will be $2 + 1 = 3$ meters away from the house. Using the Pythagorean Theorem, $x(1) = 3m$, and $c = 10m$ we calculate $h(1) = 9.54m$. Now we can simply plug everything in to our equation above and solve for $\frac{dh(1)}{dt}$:

$$2h(1)\frac{dh(1)}{dt} + 2x(1)\frac{dx(t)}{dt} = 0$$

$$2(9.54)\frac{dh(1)}{dt} + 2(3)(1) = 0$$

$$\frac{dh(1)}{dt} = \frac{-6}{19.08}$$

- Answer: $\frac{dh(1)}{dt} = -0.31$



Source: https://tutorial.math.lamar.edu/classes/calci/relatedrates.aspx

- Example Problems
  - Take the picture to the right. How fast is the shadow growing in feet per second when the person is 3 feet away from the lamp?
    - A: $x'_s = 1\ ft/s$
  - We have a cone shaped tank (point down) with water pouring in from the top. The tank is 15 ft tall and 5 ft wide and water is pouring in at a rate of 3 ft^3 per minute. How fast is the water rising per second when the water is 5 ft high? How about when the water is 10 ft high?
    - A: when h = 5, $\frac{dh}{dt} = 0.34$ ft/min
    - A: when h = 10, $\frac{dh}{dt} = 0.086$ ft/min
  - A laser pointer in set 10 meters away from a wall and is making 12 revolutions per minute. When the laser pointer is $\frac{\pi}{3}$ radians off from pointing directly at the wall, how fast is the dot from the laser pointer moving along the wall?
    - A: $\frac{dx}{dt} = 4.001\ m/s$
  - The length of each side of a cube is increasing at a rate of 3 inches per second. When each side is 2 inches, how fast is the volume of the cube growing per second?
    - A: $\frac{dv}{dt} = 36\ in^3/sec$
    - A: $\frac{dSA}{dt} = 72\ in^2/sec$
  - The height and radius of a cylinder are growing at a rate of 6 cm per minute and 4 cm per minute respectively. How fast is the volume of the cylinder increasing when the height is 20 cm and the radius is 5 cm? How fast is the surface area increasing?
    - A: $\frac{dv}{dt} = 2{,}984.51\ cm^3/min$
    - A: $\frac{dSA}{dt} = 942.48\ cm^2/min$

Things Students Tend to Struggle With

- **Related Rates**
  - Related Rates are universally considered one of the most difficult sections in Calculus 1.
  - Like most difficult problems in Calculus, none of the actual calculations in a Related Rates problem are difficult. However, because you need to do so many steps in the process of solving a Related Rates problem, it is easy to lose track of what you are doing and get lost in the weeds. The best way to avoid getting lost is to begin each problem by writing out in mathematical terms what it is that you are being asked to solve for. For instance, in the above ladder problem you are asked to solve for the speed the ladder is falling down the house. Mathematically, this is written as $\frac{dh}{dt}$. Identifying this right off the bat can make you life a lot easier.
  - Another problem that many students have with Related Rates questions is that they tend to require that the student actually produce the equations that they use to solve the problem (rather than just being given them). While this does add a level of difficulty, it is rare that the equations you are expected to produce are very complicated. Usually they take the form of a basic geometric figure, such as a circle or a cone.