# MTH 1322: Calculus II
# Week 6 Tutoring Resources

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Welcome Calculus II tutors and students! What a crazy week this last week has been. I hope that you all managed to stay safe during the week. This week's resource will cover improper integrals and how to find the area between two curves. For more help with these topics please schedule a 1-on-1 visit with me or another tutor. **Please visit baylor.edu/tutoring to make an appointment and to reserve a spot for the Calculus II group tutoring session every Tuesday at 6:30pm.** If you would like to view any of the previous resources please click **HERE**.

**Overview**[1]



**KEYWORDS:** Improper Integrals / Area between two curves

## 1 New Topics

### 1.1 Improper Integrals

So far we've been working with integrals with the bounds of integration are finite. Now we want to consider the possibility of having $\pm\infty$ as a bound of integration. We want to consider how to evaluate:

$$\int_a^\infty f(x)\,dx \text{ or } \int_{-\infty}^b f(x)dx \tag{1}$$

where $a$ and $b$ are both real numbers. To evaluate this integral we need to use a concept taught in Calculus I: limits. Using limits we can now rewrite equation (1) as the following integral [1]

$$\int_a^\infty f(x)\,dx = \lim_{R\longrightarrow\infty}\int_a^R f(x)\,dx \tag{2}$$

Similarly we might also have [1]:

$$\int_{-\infty}^b f(x)\,dx = \lim_{R\longrightarrow-\infty}\int_R^b f(x)\,dx \tag{3}$$

Recall that if the limit as $R$ approaches plus or minus infinity does not exist then the integral, $\int_a^b f(x)dx$, where a or b is equal to $R$, also does not exist. Let's look at other important properties of improper integrals. Consider we need to determine if the following integral converges or diverges:

$$\int_1^\infty \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx \tag{4}$$

[1] The information used to create this resource was taken from this textbook: [1]

Our first step should be to apply equation (1) to our integral, so that we can have:

$$\lim_{R \longrightarrow \infty} \int_1^R \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx \tag{5}$$

Now that we have a "finite" integral, we can now evaluate it using basic integration techniques. Looking closely at the integral we can see that we can use u-substitution to evaluate it. If we let $u = \sqrt{x}$ then $du = \frac{1}{2\sqrt{x}}dx$ or equivalently we have $2du = \frac{1}{\sqrt{x}}dx$. Therefore we now have the following:

$$\lim_{R \longrightarrow \infty} \int_1^R 2e^u\, du = \lim_{R \longrightarrow \infty} 2e^u \Big|_1^R = \lim_{R \longrightarrow \infty} 2e^{\sqrt{x}} \Big|_1^R \tag{6}$$

Next, we want evaluate the limit as $R$ goes to infinity. From Calculus I we know that the limit of $\sqrt{x}$ as $R$ approaches infinity is equal to infinity. Therefore we have:

$$\lim_{R \longrightarrow \infty} 2e^{\sqrt{x}} \Big|_1^R = \lim_{R \longrightarrow \infty} 2e^{\sqrt{R}} - 2e = \infty \tag{7}$$

Thus the final solution to our original problem is:

$$\int_1^\infty \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = \infty \quad \text{(diverges)} \tag{8}$$

When dealing with improper integrals it is important to remember the following formulas. Following these rules will tell us if the integral in consideration will converge or diverge. For our first formula, if we have $a > 0$ and $p > 1$ then

$$\int_a^\infty \frac{dx}{x^p} \tag{9}$$

converges and converges to $\frac{a^{1-p}}{p-1}$. However, if $p \leq 1$ then the integral diverges. Similarly, let's consider the following integral [1]:

$$\int_0^a \frac{dx}{x^p} \tag{10}$$

Even though this integral looks similar to our previous one, the rules for convergence are in fact the opposite. If we have $a > 0$ and $p < 1$ then the integral converges to $\frac{a^{1-p}}{1-p}$. However, if $p \geq 1$ then the integral diverges. Improper integrals require lots of practice to fully understand and apply ideas, that being said I recommend watching a short video to help refresh your memory. To watch the video please click **HERE** [2].

## 1.2 Area Between Two Curves

Finding the area between two curves follows concepts very similar to volumes of revolution. The general formula we use to find the area between two curves is the following equation [1]:

$$A = \int_b^a (y_{\text{top}} - y_{\text{bot}})dx = \int_b^a (f(x) - g(x))dx \tag{11}$$

where $A$ is the area we want to find, $y_{\text{top}}$ is the function that is graphically above the other function which we denote as $y_{\text{bot}}$. Let's work an example to have a better understanding of how to apply the formula. Suppose we are asked to find the area between the function $f(x) = x^2 + 2x + 1$ and $g(x) = 2x + 2$. If were to graph these two function we could easily see which one is the top function is the bottom function and then apply the formula. So let's try to work this one without relying of the visualization of the graph. Our first step is to find where these two functions are equal to each other so we know our bounds of integration.

$$x^2 + 2x + 1 = 2x + 2 \longrightarrow x^2 - 1 = 0 \tag{12}$$

From here we can see that the two curves will intersect at $\pm 1$ by finding the roots of the polynomial in the previous equation: $x^2 = 1$. These roots tell us our bounds of integration so now we need to determine which function we consider our top function and which function we consider our bottom function. To do so we need to evaluate each function at some point between our bounds integration. We will use 0 since not only it is the easiest $x$ value to evaluate but also since it lies between -1 and 1. As we see $f(0) = 1$ and $g(0) = 2$ thus we can apply the formula:

$$A = \int_{-1}^{1} ((2x+2) - (x^2+2x+1))\, dx \longrightarrow \int_{-1}^{1} (-x^2-1)\, dx \tag{13}$$

Which by basic rules on integration we find the answer to be $-2\frac{2}{3}$.

While we solved a relatively simple example it is important to understand there are some exceptions to our method. I recommend watching a short video to familiarize yourself with some other examples. To watch a short video about this concept click **HERE** [3].

## References


[1] J. Rogawski, C. Adams, and R. Franzosa, *Calculus: Early Transcendentals*, 4th ed. New York: W. H. Freeman, Dec. 2018.

[2] “Introduction to improper integrals (video).” [Online]. Available: https://www.khanacademy.org/math/ap-calculus-bc/bc-integration-new/bc-6-13/v/introduction-to-improper-integrals

[3] “Area between curves | Applications of definite integrals | AP Calculus AB | Khan Academy.” [Online]. Available: https://www.youtube.com/watch?v=wQXYtsyfbqg