MTH 1322: Calculus II
Week 4 Tutoring Resources

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Welcome Calculus II tutors and students! In the upcoming week there are two sections that have exams so be prepared to see an increase in Cal II traffic. **Please visit** baylor.edu/tutoring **to make an appointment and to reserve a spot for the Calculus II group tutoring session every Tuesday at 6:30pm.**

Overview[1]

KEYWORDS: volumes of revolution / partial fraction decomposition / area between two curves

1 New Topics

1.1 Average Value, Density, Volume

In this section we tackle three applications of integrals: volume, density and average value. Using integrals to find volumes of objects should feel natural since integrals find the area under a given function. In this case that is almost exactly what we are doing.

To find the volume of an object we start by first taking the area of a cross section of the object, $A(y)$. We then multiply the area by the height of the cross section, δy or equivalently dy. Therefore, if we sum every slice we can obtain the volume of the object. Thus we have the formula used to find the volume of a solid object [1]:

$$V = \int_b^a A(y)dy \tag{1}$$

As an an example; suppose we have a sphere with radius R. Our first step is to find a formula to generalize the area of each cross section. Even though our general formula for volume tells us to find the horizontal cross sections, we use the vertical cross sections since we are working with a sphere. First we see the shape of our cross section is a circle whose radius r satisfies $x^2 + r^2 = R^2$ or equivalently $r = \sqrt{x^2 + R^2}$ [1]. As you can see in the image below, the green line represent the constant radius of the sphere, R, and the blue line represents the radius of the cross sections.

[1]The information used to create this resource was taken from this textbook: [1]

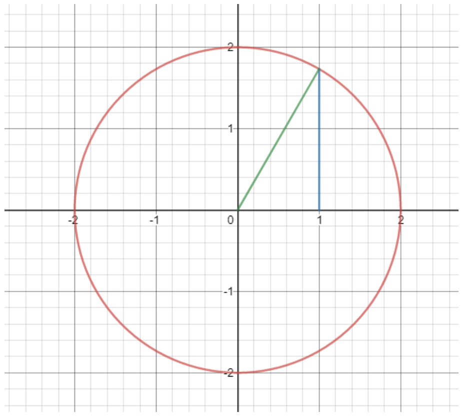

Figure 1: Sphere of Radius R [2]

Thus we see the area of each cross section is given by: $A(x) = \pi r^2 = \pi(R^2 - x^2)$. Therefore, the volume of sphere is found by:

$$\int_{-R}^{R} \pi(R^2 - x^2)dx \tag{2}$$

The other major concept that comes from this section is the Mean Value Theorem (MVT). Before we define the MVT we first look at what the average value of a function means. Recall that we define the average value of a function to be:

$$\text{Average Value} = M = \frac{1}{b-a}\int_{b}^{a} f(x)dx \tag{3}$$

The MVT for integrals states: if f is continuous on [a,b] with average value M, then $f(c) = M$, for some $c \in [a, b]$ [1].

To watch a video that provides an in-depth explanation to find volumes using integration click **HERE** [3].

1.2 Volumes of Revolution (Disk & Washer Method)

In this section we discuss how to find the volume of a solid that was formed by rotating a region of the plane about an axis. Disk and washer method get their names from the type of object we are finding the volume of. If f is continuous and $f(x) \geq 0$ on [a,b], then the solid obtained by rotating the region under the graph about the x-axis has volume given by the general equation for disk method: [1]

$$V = \pi\int_{a}^{b} R^2 dx = \pi\int_{a}^{b} f(x)^2 dx \tag{4}$$

As an example, suppose we are asked to find the volume of $f(x) = (x - 2)^2 + 4$ from $x = 0$ to $x = 2$ rotated about the x-axis. Looking at the image below we can see that we are asked to find the volume of the object created by rotating the highlighted region about the x-axis. By observation we see that our $f(x)$ satisfies the conditions to use the formula for disk method.

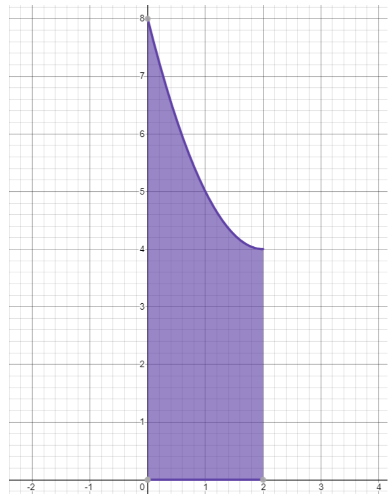

Figure 2: Region below $(x-2)^2+4$ [2]

Therefore we can say the volume of the object created is defined to be:

$$V = \pi \int_0^2 ((x-2)^2+4)^2 dx = \pi \int_0^2 (x^2-4x+8)(x^2-4x+8)\, dx = \tag{5}$$

$$= \pi \int_0^2 (x^4 - 4x^3 + 8x^2 - 4x^3 + 16x^2 - 32x + 8x^2 - 32x + 64)\, dx \tag{6}$$

$$= \pi \int_0^2 (x^4 - 8x^3 + 32x^2 - 64x + 64)\, dx \tag{7}$$

The main concern with these methods is the set up and not so much the calculation of the integral. Therefore I will refrain from showing the final step of integrating the polynomial. I recommend you work the rest of the problem out yourself to get more practice integrating. If you would like to check your work the answer to the above problem is: 187.6580

The next method we discuss, Washer method is merely a variation of Disk method. We use washer method when we are asked to find the volume of an object given by rotating the area between two curves. Suppose we have $f(x) \geq g(x) \geq 0$ where $f(x) - g(x)$ is the area between two curves; then we can define washer method to be:

$$V = \pi \int_a^b (R_{\text{outer}}^2 - R_{\text{inner}}^2) dx = \pi \int_a^b (f(x)^2 - g(x)^2) dx \tag{8}$$

Observe that if we are given functions in terms of y we could rewrite the equation to be as follows:

$$V = \pi \int_a^b (R_{\text{outer}}^2 - R_{\text{inner}}^2) dy = \pi \int_a^b (f(y)^2 - g(y)^2) dy \tag{9}$$

Let's work an example problem. Suppose we are asked to find the volume of a solid obtained by rotating the region under the graph of $f(x) = -(x-2)^2 + 4$ for $0 \leq x \leq 4$ about the vertical axis $x = 3$. The object formed would look like the image below.

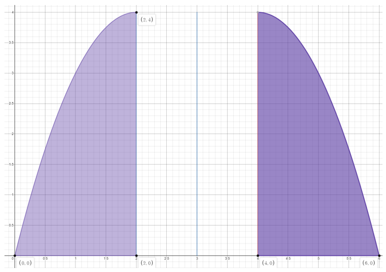

Figure 3: 9 - x^2 rotated about y-axis [2]

Since we are rotating about the y-axis and using washer method, we will integrating with respect to y. Therefore we need to convert from terms of x into terms of y. Thus we have:

$$y = 4 - (x-2)^2 \longrightarrow (x-2)^2 = 4 - y \longrightarrow x = \pm(\sqrt{4-y} + 2) \tag{10}$$

Now lets determine what the inner and outer radii will be. As a general rule, the radii for both washer and disk are always perpendicular to the axis of rotation. Therefore, we see our outer radius is $\sqrt{4-y}+2$ and our inner radius is 1; since we are rotating about the vertical line of $x = 3$. Thus we can find the volume by solving the following integral:

$$V = \pi \int_0^4 ((\sqrt{4-y}+2)^2 - 1)dy \tag{11}$$

The exact solution can be found by solving the integral which will can simply be solved by hand or calculator so we will not solve for the exact solution.

Khan Academy provides great videos with some great illustration to better explain both Disk and Washer method. To watch the video series on Disk method click **HERE** [4]. To watch the series on Washer method click **HERE** [5].

1.3 Volumes of Revolution (Shell Method)

The last method we learn to find volume of an object given by a rotated region is Shell method. The main notable difference between shell method and disk / washer method is for shell method, the shell height is necessary to find the volume which is always parallel to the axis of rotation. Whereas, the radii for disk and washer method necessary to find the volume are always perpendicular to the axis of rotation [1]. Recall that the formula for shell method for an object rotated about the y-axis is the following equation

$$V = 2\pi \int_a^b (\text{radius})(\text{height of shell})dx \tag{12}$$

Let's look at an example: Suppose we have $f(x) = -(x-5)^2 + 9$ and $g(x) = (x-3)^2 + 5$ and we are asked to find the volume of region $f(x) - g(x)$ rotated about the y-axis.

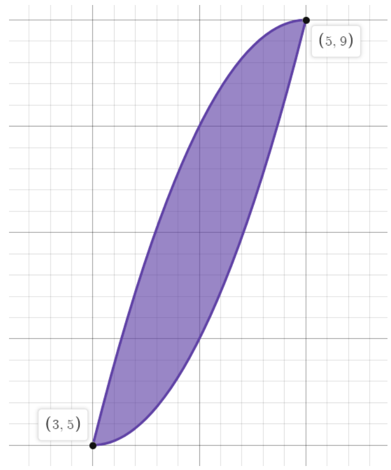

Figure 4: $f(x) - g(x)$ [2]

Using equation (12) and the figure above we can find the radius and height of the shell we are integrating. We can see the height of the shell is determined by $f(x) - g(x)$. Furthermore we see the radius of the shell changes based on our x value, thus our radius is x. Therefore the volume of the object given by rotating the region about the y-axis can be found by the following integral:

$$v = 2\pi \int_1^4 x[f(x) - g(x)]dx = 2\pi \int_1^4 x[(-(x-5)^2 + 9) - ((x-3)^2 + 5)]dx \tag{13}$$

Again since the solution to this integral can be found by plugging by using a calculator we will not work the full solution.

Khan Academy also provides a video series explaining shell method with great illustrations. To watch the series click **HERE** [6].

1.4 Method of Patial Fractions

The last topic we will review is Method of Partial Fractions. Partial fractions is used when we have integrals that involve polynomials being divided by other polynomials that cannot be simplified. For example, consider the following integral [1]:

$$\int \frac{x^2 + 2}{x(x^2 - 25)} dx \tag{14}$$

It is obvious that none of our other tricks such as trig-substitution or u-substitution will work to solve this integral, so we must use method of partial fractions. To solve this let's ignore the integral for right now and focus on the partial fraction decomposition. Notice that if we factor the denominator it will easier to see how to apply partial fraction decomposition

$$\frac{x^2 + 2}{x(x^2 - 25)} = \frac{x^2 + 2}{x(x+5)(x-5)} = \frac{A}{x} + \frac{B}{x-5} + \frac{C}{x+5} \tag{15}$$

Next we multiply by $x(x+5)(x-5)$ to clear the denominators to get

$$x^2 + 2 = A(x-5)(x+5) + B(x)(x+5) + C(x)(x-5) \tag{16}$$

Now our goal is find the value of each of the constants. I recommend choosing values of x that will make each constant easier to solve for. For example, if we let $x = 5$:

$$5^2 + 2 = A(5-5)(5+5) + B(5)(5+5) + C(5)(5-5) \longrightarrow 27 = 0 + B(50) + 0 \tag{17}$$

we see that $B = \frac{27}{50}$. Notice that carefully choosing our x values simplifies the amount of work we need to do. So if we let $x = -5$ we have:

$$27 = A(-5-5)(-5+5) + B(-5)(-5+5) + C(-5)(-5-5) \longrightarrow 27 = -50C \longrightarrow C = -\frac{27}{50} \tag{18}$$

Next we can see that letting $x = 0$ gives us:

$$2 = A(5)(-5) + B(0)(5) + C(0)(-5) \longrightarrow 2 = A(-25) \longrightarrow A = -\frac{2}{25} \tag{19}$$

Therefore since we have solved for each constant we can say

$$\frac{x^2+2}{x(x^2-25)} = \frac{-\frac{2}{25}}{x} + \frac{\frac{27}{50}}{x-5} + \frac{-\frac{27}{50}}{x+5} \tag{20}$$

Therefore we can rewrite (14) to be the following

$$\int \frac{x^2+2}{x(x^2-25)} dx = -\frac{2}{25}\int \frac{dx}{x} + \frac{27}{50}\int \frac{dx}{x-5} - \frac{27}{50}\int \frac{dx}{x+5} \tag{21}$$

Evaluating each integral separately we can see the solution is:

$$-\frac{2}{25}\ln|x| + \frac{27}{50}\ln|x-5| - \frac{27}{50}\ln|x+5| + C \tag{22}$$

Since they are too many special cases of partial fractions to cover in this in this resource I would highly recommend watching one of the videos that I have linked. To watch a short video explaining Partial Fractions click **HERE** [7]. If you have time, I also recommend watching this longer video that works more example problems involving partial fractions (**HERE**) [8].

2 Topics Previously Covered

2.1 Weeks 1 & 2

In weeks 1 & 2 we covered Trigonometric Substitution, U-Substitution, Trigonometric Integrals, and Integration by parts. If you would like to view this Resource please click **HERE**. As the semester progresses, I will continue to list out the which topics were covered in what week, as well as link directly to the resource.

References

[1] J. Rogawski, C. Adams, and R. Franzosa, *Calculus: Early Transcendentals*, 4th ed. New York: W. H. Freeman, Dec. 2018.

[2] “Desmos | Graphing Calculator.” [Online]. Available: https://www.desmos.com/calculator

[3] “AP Lesson 6.2 Volume, Density, and Average Value (Part 1),” Mar. 2013. [Online]. Available: https://www.youtube.com/watch?v=tCOCQPHzIn8

[4] “Disc method around x-axis (video).” [Online]. Available: https://www.khanacademy.org/math/ap-calculus-ab/ab-applications-of-integration-new/ab-8-9/v/disk-method-around-x-axis

[5] “Solid of revolution between two functions (leading up to the washer method) (video).” [Online]. Available: https://www.khanacademy.org/math/ap-calculus-ab/ab-applications-of-integration-new/ab-8-11/v/disc-method-washer-method-for-rotation-around-x-axis

[6] “Volume using calculus | Integral Calculus (2017 edition).” [Online]. Available: https://www.khanacademy.org/math/old-integral-calculus/volume-using-calculus-ic

[7] “Integration with partial fractions (practice).” [Online]. Available: https://www.khanacademy.org/math/ap-calculus-bc/bc-integration-new/bc-6-12/e/integration-of-rational-functions-by-division-and-partial-fractions

[8] “Integration By Partial Fractions.” [Online]. Available: https://www.youtube.com/watch?v=r5MIraVUVUA&t=982s