MTH 1322: Calculus II
Week 1-2 Tutoring Resources

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Welcome Calculus II tutors & students! My name is Ethan Reyes and I will be the master tutor for Calculus II this semester. I am a senior mathematics major and I look forward to working with you all this semester! I hope that everyone has had a great first two weeks and is staying safe! This resource will review the topics covered in the first two weeks it will be longer than usual. **Please visit baylor.edu/tutoring to make an appointment and to reserve a spot for the Calculus II group tutoring session every Tuesday at 6:30pm.**

Overview[1]

KEYWORDS: indefinite integral / trigonometric identities / integration by parts

1 Concepts from weeks 1 & 2

1.1 U-Substitution

The first concept covered by all sections is u-substitution, a topic typically taught in the last weeks of Calculus I. Let's consider the following integral

$$\int x \sin x^2 dx \tag{1}$$

We want to let $u = x^2$ so that after differentiating both sides we have: $\frac{du}{dx} = 2x$. Now, if we multiplyt both sides by $\frac{1}{2}dx$ we get the following equation $\frac{1}{2}du = xdx$. Now we can rewrite the integral as:

$$\int \frac{1}{2} \sin u du = \frac{1}{2} \int \sin u du \tag{2}$$

Now, since the integral of $\sin u = -\cos u$ we now write the solution in terms of u:

$$-\frac{1}{2} \cos u + c \tag{3}$$

Always remember to include the "+c" since we are dealing with an indefinite integral. Since $u = x^2$ we substitute x^2 in for u. Which gives us the solution to equation (1):

$$\boxed{\int x \sin x^2 dx = -\frac{1}{2} \cos x^2 + c} \tag{4}$$

We can double check that this is the solution by taking the derivative using chain rule. Click **HERE** to watch a short video to learn more about u-substitution [2].

[1] The information used to create this resource was taken from this textbook: [1]

1.2 Integration By Parts

Next we discuss integration by parts; the first "true" concept of Calculus II. Consider the following integral,

$$\int xe^x dx \tag{5}$$

observe that we cannot use u-substitution nor can we use normal integration to solve it. Recall the formula for integration by parts [1]:

$$\int udv = uv - \int vdu \tag{6}$$

For those more comfortable/familiar with $f(x)$ notation, we can rewrite equation (6) as the following:

$$\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x) \tag{7}$$

As aforementioned, equation (6) and equation (7) are equivalent to one another but simply use different notation. To watch a short video of the derivation, click HERE [3]. Notice how we are integrating one function and then taking the derivative of the other. It is important to know how to choose which function takes on the derivative and which one takes on the integration. Notice that if we were have:

$$\int x \sin x \, dx \tag{8}$$

we could choose to differentiate $\sin x$ and integrate x but notice that this would give us a more confusing integral to deal with. The general idea of integration by parts is we are "passing" the derivative from one function to the other as seen in equation (7). Let's look at an example to make this more clear. Consider the following integral:

$$\int \ln x \, dx \tag{9}$$

Our first step is to break equation (9) into two different functions so that we can apply equation (7). Notice that at first glance there only appears to be 1 function in the integral. However, we can let $g(x) = 1$ be the second function and let $f(x) = \ln x$. Notice that we can now apply equation (7) i.e. we want to integrate $g(x) = 1$ and differentiate $f(x) = \ln x$. Doing so would yield the following equation:

$$\int \ln x \, dx = x \ln x - \int \frac{1}{x} x \, dx = x \ln x - \int 1 \, dx \tag{10}$$

We know that the indefinite integral of 1 is equal to $x + c$. Thus our final solution to our original problem is:

$$\int \ln x \, dx = x \ln (x) - x + c = x(\ln x - 1) + c \tag{11}$$

There are other ways to solve this problem besides using equations (7) and (6). Another common approach to solving these types of problems is by using tabular method. Click **HERE** [4] for a video explaining how to approach integration by parts using the tabular method.

1.3 Trigonometric Integration

Trigonometric integration heavily relies on trigonometric identities taught in Pre-Calc. The key to this topic is being able to recognize when to apply certain trigonometric identities. The integrals covered in this section start with the base form of:

$$\int \sin^n x \cos^m x dx \tag{12}$$

Let's consider the case of odd powers of $\sin x$; i.e. Let's consider the following integral:

$$\int \sin^3 x dx \tag{13}$$

Now, using the the following trigonometric identity we have $\sin^2 x = 1 - \cos^2 x$. Therefore, we can rewrite our integral to be $\int (1 - \cos^2 x)(\sin x)dx$. Thus,

$$\int (1 - \cos^2 x)(\sin x)dx = \int \sin x dx - \int \sin x \cos x dx = \frac{1}{2}\cos^2 x + c \tag{14}$$

Now we examine odd powers of $\cos x$. Looking at the example below,

$$\int \sin^4 x \cos^5 x dx \tag{15}$$

we see that we can rewrite $\cos^5 x$ as $(\cos^4 x)(\cos x)$. Since $\cos^2 x = 1 - \sin^2 x$ it follows that $\cos^4 x = (1 - \sin^2 x)^2$. Thus our new integral becomes,

$$\int (\sin^4 x)(\cos x)(1 - \sin^2 x)^2 dx = \int (\sin^4 x - \sin^6 x + \sin^8 x)\cos x dx \tag{16}$$

Now using u-substitution we can set $u = \sin x$, so then $du = \cos x$, therefore we have

$$\begin{aligned}\int (u^4 + u^8 - u^6)du &= \frac{u^5}{5} + \frac{u^9}{9} - \frac{u^7}{7} + C \\ &= \boxed{\frac{sin^5 x}{5} + \frac{\sin^9 x}{9} - \frac{\sin^7 x}{7} + C}\end{aligned} \tag{17}$$

Students often struggle with remembering trig-identities and recognizing where to apply those identities. To watch a video that dives into greater detail click **HERE** [5].

1.4 Trigonometric Substitution

Trig substitution combines the ideas of trig-integration and u-substitution. There are three formulas that are covered, the first being if we have $\sqrt{a^2 - x^2}$ in our integral we let $x = a\sin\theta$ so that we have $dx = \cos\theta$. Let's solve the integral below as an example;

$$\int \frac{1}{\sqrt{1 - x^2}} dx \tag{18}$$

now we apply our formula from above to get

$$\int \frac{1}{\sqrt{1 - \sin^2\theta}} \cos\theta d\theta \tag{19}$$

if we look closely we see that the $1 - \sin^2\theta$ is a trig-identity that is equal to $\cos^2\theta$. Therefore we can rewrite the integral

$$\int \frac{\cos\theta d\theta}{\sqrt{\cos^2\theta}} = \int \frac{\cos\theta d\theta}{\cos\theta} = \int d\theta \tag{20}$$

Therefore the solution is $\theta + c$, but since we need the answer in terms of x we must solve $x = \sin\theta$ for θ. Solving for θ yields $\theta = \arcsin x$, which means our final solution is:

$$\boxed{\frac{1}{\sqrt{1 - x^2}} dx = \arccos x + c = \cos^{-1} + c} \tag{21}$$

Next we consider the case where we have $\sqrt{x^2 - a^2}$ in the integrand. Recall that if we have $\sqrt{x^2 - a^2}$ then we need to let $x = a\sec\theta$ so that $dx = a\sec\theta\tan\theta d\theta$. Let's work an example:

$$\int \frac{dx}{x^x\sqrt{x^2 - 9}} \tag{22}$$

since we see that $a = 3$ we know that we can use $x = 3\sec\theta$ which allows to also use $dx = 3\sec\theta\tan\theta$ rewrite the integral as follows;

$$\int \frac{3\sec\theta\tan\theta d\theta}{(3\sec\theta)^2\sqrt{(3\sec\theta)^2 - 3^2}} = \int \frac{3\sec\theta\tan\theta d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta - 9}} \tag{23}$$

Now we can apply the trig-identity $\sec^2\theta - 1 = \tan^2\theta$ to rewrite the integral again as:

$$\int \frac{3\sec\theta\tan\theta d\theta}{9\sec^2\theta\sqrt{9\tan^2\theta}} = \int \frac{3\sec\theta\tan\theta d\theta}{9\sec^2\theta 3\tan\theta} = \int \frac{d\theta}{9\sec\theta} \tag{24}$$

but since we know that $\frac{1}{\sec\theta} = \cos\theta$ it follows that we can rewrite the integral again:

$$\int \frac{d\theta}{9\sec\theta} = \frac{1}{9}\int \cos\theta d\theta = \frac{1}{9}\sin\theta + c \tag{25}$$

Since $x = 3\sec\theta \implies \sec\theta = \frac{x}{3}$. Notice that $\sec x = \frac{1}{\cos x} = \frac{x}{3}$. Thus we can say that $\cos x = \frac{3}{x} = \frac{\text{adjacent}}{\textit{hypotenuse}}$. Therefore, we can now set up a triangle where the hypotenuse is equal to x, the adjacent side is equal to 3 which means our opposite is equal to $\sqrt{x^2 - 9}$. Since $\sin\theta = \frac{\text{opposite}}{\textit{hypotenuse}} = \frac{\sqrt{x^2-9}}{x}$ it follows that $\frac{1}{9}\sin\theta = \frac{\sqrt{x^2-9}}{9x}$. Therefore the final answer to our initial integral is the following:

$$\int \frac{dx}{x^x\sqrt{x^2 - 9}} = \frac{1}{9}\sin\theta + c = \frac{\sqrt{x^2 - 9}}{9x} + c \tag{26}$$

The last formula we consider is when $\sqrt{x^2 + a^2}$ so that we can let $x = a\tan\theta$ which gives us $dx = a\sec^2\theta d\theta$. For example:

$$\int \frac{1}{\sqrt{x^2 + 9}} dx \tag{27}$$

we see that if we let $x = 3\tan\theta$ then we can rewrite the integral as:

$$\int \frac{3\sec^2\theta d\theta}{\sqrt{(3\tan\theta)^2 + 3^2}} = \int \frac{3\sec^2\theta d\theta}{\sqrt{9\tan\theta^2 + 9}} \tag{28}$$

using trig-identities we know that $9\tan^2\theta + 9 = 9\sec^2\theta$ therefore we have:

$$\int \frac{3\sec^2\theta d\theta}{\sqrt{9\tan\theta^2 + 9}} = \int \frac{3\sec^2\theta d\theta}{\sqrt{9\sec^2\theta}} = \int \frac{3\sec^2\theta d\theta}{3\sec\theta} = \int 3\sec\theta d\theta = \ln|\sec\theta + \tan\theta| + c \tag{29}$$

Once again we need our answer in terms of x instead of θ. Since $x = 3\tan\theta$ it follows that $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{3}$ and furthermore we know that the hypotenuse is equal to $\sqrt{x^2 + 9}$. Therefore we can use this triangle to help us change variables. Since $\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$ we know that $\sec\theta = \frac{\sqrt{x^2+9}}{3}$. Which gives us our final solution:

$$\ln\left|\frac{\sqrt{x^2 + 9}}{3} + \frac{x}{3}\right| + C \tag{30}$$

Students most often have trouble with converting their answer from terms of θ into terms of x. If possible I would recommend attempting to draw a triangle to explain how to convert from θ to x. To watch a video that covers Trig-Substitution click **HERE** [6].

References

[1] J. Rogawski, C. Adams, and R. Franzosa, *Calculus: Early Transcendentals*, 4th ed. New York: W. H. Freeman, Dec. 2018.

[2] K. Academy, “u-substitution intro,” Youtube, 2012. [Online]. Available: https://www.youtube.com/watch?v=7hCsQOKOYS8

[3] Learn Math Tutorials, “What is Integration by Parts - How to do Integration by Parts,” Feb. 2013. [Online]. Available: https://www.youtube.com/watch?v=bLhxQIdbWW8

[4] T. O. C. Tutor, “Integration by parts - tabular method,” Youtube, 2018. [Online]. Available: https://www.youtube.com/watch?v=Yyic5aaXGaw

[5] PatrickJMT, “Trigonometric integrals- part 1,” Youtube, 2008. [Online]. Available: https://www.youtube.com/watch?v=lTqnlihOC4o

[6] K. Academy, “Introduction to trigonometric substitution,” Youtube, 2015. [Online]. Available: https://www.youtube.com/watch?v=EV5dhv0A2wU