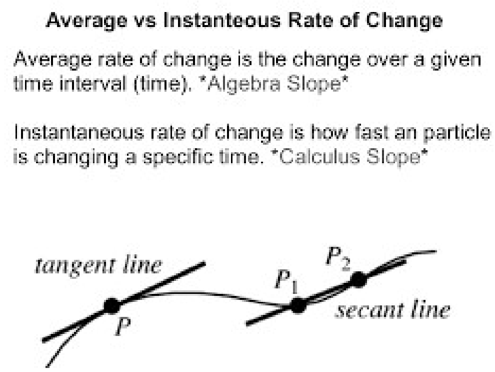
Calculus 1, Week 3

Hey Calculus tutors and students! My name is Alex Williamson, and I am a University Scholars major concentrating in Economics and Math. I am also the Master Tutor for Calculus 1 at the Tutoring Center here on campus. This is the first of many resources here to help explain Calculus 1. This first resource covers the first three weeks of class, so I apologize for the length, but future resources should be much shorter. Specifically, in this resource I will cover how to estimate the slope of the Tangent line, how to identify the three different kinds of discontinuities, how to evaluate limits, and how to calculate derivatives at a point using limits.

In addition to this resource and our one-on-one tutoring sessions, the Baylor Success Center also runs a weekly group tutoring session for Calculus 1. The group tutoring session occurs every Wednesday from 5pm-6pm. If you are interested in attending you can reserve a spot through our website: https://www.baylor.edu/support_programs/index.php?id=40917

As with last year, if anyone has any suggestions for things to add to the resource please do not hesitate to contact me!

- Alex Williamson



Source: https://mcdowellakmath.weebly.com/uploads/2/4/5/4/24545915/1_velocity_and_rates_of_change.pdf

Keywords: Limits, Continuity, Discontinuity, IVT, Derivatives.

Key

- Yellow Highlighting: Definitions that you need to know.
- Green Highlighting: Explanation of how you actually go about doing the problems.

Concepts

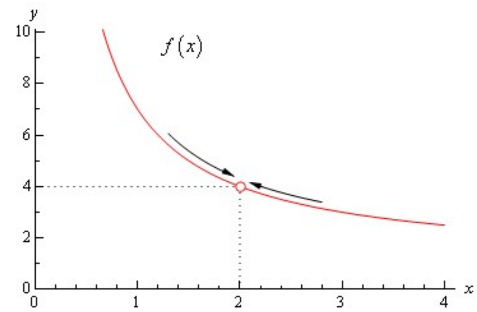
- **2.1: Limits, Rates of Change, and Tangent Lines**
  - Average Rate of Change
    - The average rate of change is a measure of how fast on average an object is moving across a period of time. For instance, if I wanted to calculate what my average rate of change (or average speed) while driving from Waco to Dallas, I would look at the total amount of time I took to get to Dallas, divided by the length of the route that I took. The number that I get would end up being the average rate of change of my position relative to Dallas (the speed I was driving) during my drive. Now, that doesn't mean that I was driving this speed the whole time – sometimes I go caught in traffic, and other times I was along and could speed up – but this is my average speed across my whole drive. The average rate of change from point one to point two is the same as the slope of the Secant line from point one to point two, as seen in the graphic above.
  - Calculating Instantaneous Rate of Change from Average Rate of Change
    - While calculating the average speed I drove while on the way to Dallas can be helpful, I may also want to calculate the specific speed that I was going at some

random point on my drive. When doing this I would be calculating the instantaneous rate of change (speed at one moment in particular), rather than the average rate of change (my average speed over the whole drive). One way we can estimate this instantaneous rate of change without using derivatives is by calculating the average rate of change over a short interval. So if I want to estimate the instantaneous rate of change at t=3, I might calculate the average rate of change from t=3 to t=3.01.

- Video
  - https://www.youtube.com/watch?v=0T5y7FaJzJU&t=5s
- Example
  - If it takes me an hour and a half to drive the 100 miles to Dallas, what was my average rate of change (average speed) while driving?
    - A: 66.67 miles per hour
  - Estimate the instantaneous rate of change of B with respect to T when T=20 and B=$7\sqrt{T}$
    - A: 0.7826

- **2.2: Limits: A Numerical and Graphical Approach**
  - Limits
    - The purpose of a limit is to describe the behavior of a function as it goes towards a specific value of X (or any other variable). The simplest way of finding a limit is to graph the function in question and see where it is going as X approaches the designated value from both sides. By plotting points closer and closer to the desired point, we can get a better and better idea of what the function is doing at that point

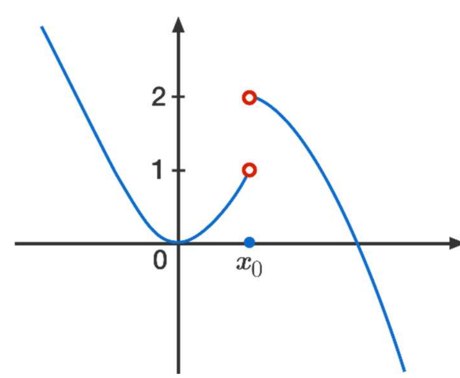
A Two-sided, or regular Limit



Source: https://tutorial.math.lamar.edu/classes/calci/thelimit.aspx

  - One-sided Limits
    - Sometimes the limit of a function may be different depending on which side you come from. In this case, no two-sided (regular) limit exists, but either of the two one-sided limits might still exist.
  - Examples
    - Evaluate $\lim_{x\to -1} \frac{x^2+2x+1}{x+1}$
      - A: 0
    - What are the right- and left-hand limits of $\lim_{x\to 0} \frac{1}{x}$
      - A: the left-hand limit goes to negative infinity while the right-hand limit goes to positive infinity.

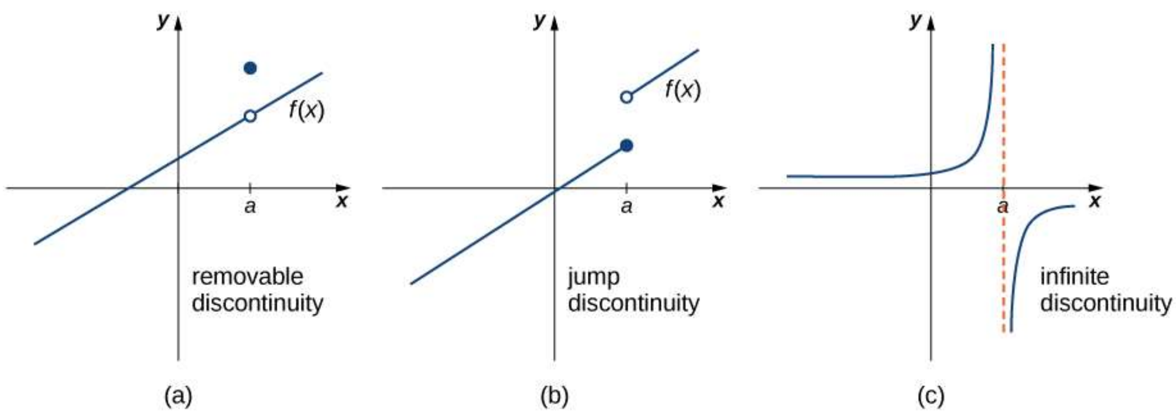
One-sided Limits



Source: https://brilliant.org/wiki/when-does-a-limit-exist/

- **2.3: Basic Limit Laws**
  - There are several ways that we can manipulate a limit equation in order to make it easier to solve. Several examples are laid out in your book, such as the Sum Law, Constant Multiple Law, and more. This video goes through several of those laws.
  - Video

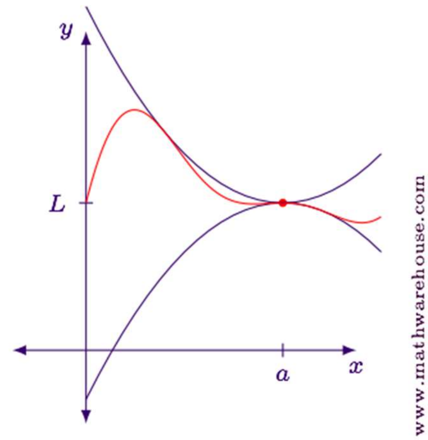
    - https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-new/ab-1-5a/v/limit-properties
  - Example
    - Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^5$
      - A: 1
- **2.4: Limits and Continuity**



Source: https://math.libretexts.org/Courses/Misericordia_University/MTH_171-172%3A_Calculus_-_Early_Transcendentals_(Stewart)/02%3A_Limits_and_Derivatives/2.05%3A_Continuity

  - Discontinuity types
    - There are several discrete types of discontinuity that can occur in a function, specifically the removable, jump, and infinite discontinuity types.
  - Continuity at a point
    - A function is continuous at a point if the two-sided limit of the function exists at that point and equals the actual value of the function at that point. A function is continuous in general if it is continuous at every point.
  - Examples
    - What type of discontinuity is $\frac{1}{x}$?
      - A: infinite
    - Is the function $\frac{1}{x}$ continuous at x = 1?
      - A: yes
- **2.5: Evaluating Limits Algebraically**
  - Many times in calculus we will be faced with a limit which, if solved in a certain way, will result in an indeterminate form (zero over zero or infinity over infinity). While we will eventually learn L'Hospital's Rule to solve these sorts of problems, for now **WE ARE NOT ALLOWED TO USE IT IN CALC 1**. The best thing to do is to use rules of algebra to put this derivative in a form that we can find an actual answer for.
  - Example
    - What is $\lim_{x\to-1}\left(\frac{x+1}{x^2-1}\right)^2$
      - A: $\frac{1}{4}$
- **2.6: The Squeeze Theorem and Trigonometric Limits**

- In this section we are tasked with finding the limit of several functions that contain trig. Unfortunately, the many of these functions are very difficult to do directly. Instead, we will use the Squeeze Theorem to solve them.
- Video
  - https://www.youtube.com/watch?v=js45kis2ZoI&t=2s
- Example
  - What is the $\lim_{x \to 0} x\, sin\left(\frac{1}{x}\right)$?
    - A: 0



The middle function is squeezed to L as x approaches a.

- **2.7: Limits at Infinity**
  - In addition to using a limit to describe the behavior of a function on the part of the graph we can see, we can also use limits to describe the behavior of a function as it heads to infinity or negative infinity. To calculate this limit we can look at asymptotes, as well as just the general behavior of the function.
  - Example
    - What is the $\lim_{x \to \infty} \frac{70x^3+100x^2}{5x^4+35x}$?
      - A: 0
- **2.8: The Intermediate Value Theorem**
  - The textbook definition of IVT is fairly complex, and while taking Calc 1 you will need to know it, but in simple English, this theorem states that if a function is continuous, then it can't "skip" y values that are between its start and end values of y (which are f(a) and f(b)).
  - Example
    - Prove that x^3 has a root on the interval [-1,1]
      - A: since f(-1) is negative, and f(1) is positive, by the IVT it must equal zero (a point between them) at some point "c"
- **3.1: Definition of the Derivative**
  - A derivative of a function is the same as its instantaneous rate of change from Ch. 2, which means that it is also the same as the slope of the tangent line of a function at a given point. While you could always find a derivate by graphing the function, and then estimating the tangent line, it is much

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x \to a}$$

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easier to find the derivative using its definitions.

        - Before when we estimated the slope of the tangent line we created a table and slowly got closer and closer to the point we actually wanted. Now we are doing the exact same think, but with limits.

    - Video
        - https://www.youtube.com/watch?v=DxzTXfGHRDo
    - Examples
        - What is the derivative of f(x) = x^2 at x = 3?
            - A: $f\,'(3) = 6$
        - Find the Derivative of $f(x) = 3x^2 + 7x + 10$ at $x = 4$
            - A: $f\,'(4) = 31$

Things Students Tend to Struggle With

- **Indeterminant Limits**
    - If you get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when evaluating a limit, your next step should be to look for terms that can cancel in the numerator and denominator.
- **Squeeze Theorem**
    - The key to most squeeze theorem problems is finding the part of the original equation that is always bounded (usually $sin(x)$ and $cos(x)$) and then recreating your original function within the bounds of the inequality you identified.
- **Limits at Infinity**
    - The big takeaway from limits at positive or negative infinity is that only the largest terms in the numerator and denominator matter.
- **Limit Definition of the Derivative**
    - When using the $f\,'(x) = \lim_{h->0} \frac{f(x+h)-f(x)}{h}$ version of the limit definition of the derivative, the whole goal is to pull an "h" out of the numerator to cancel with the one in the denominator. For $f\,'(x) = \lim_{x->a} \frac{f(x)-f(a)}{x-a}$ the same is true, except what you really need to cancel is the "$x - a$" term.