MTH 2311 Linear Algebra
Week 9 Resources

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Major Topics:

1. The Four Fundamental Subspaces (Continued)
2. Eigenvalues and Eigenvectors

Textbook Material:

Linear Algebra and Its Applications, 5th Edition by Lay and McDonald
Sections 4.2-4.4

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1 Conceptual Review

1.1 The Four Fundamental Subspaces (Continued)

Last week we introduced the notion of the four fundamental subspaces and how to solve for them. Recall that the *left null space*, is not mentioned until the final chapters of the book, so you should only really be aware of the fact that it exists and is the same thing as simply the null space of the transpose of a matrix. As a brief review, the four fundamental subspaces are:

Suppose we have a matrix $\mathbf{A} \in \mathbb{R}^m \times n$ (that is, $\mathbf{A}$ has m rows and n columns). The definitions of the four fundamental subspaces of $\mathbf{A}$ are as follows:

(i) The *column space* of $\mathbf{A}$ is the span of the columns of $\mathbf{A}$, which is the same thing as the range of $\mathbf{A}$. The *row space* of $\mathbf{A}$ is a subspace of $\mathbb{R}^m$.

(ii) The *row space* of $\mathbf{A}$ is the span of the rows of $\mathbf{A}$. Upon closer inspection, this is the same thing as the span of the columns of $\mathbf{A}^T$, which is the same thing as the range of $\mathbf{A}^T$. The *row space* of $\mathbf{A}$ is a subspace of $\mathbb{R}^n$.

(iii) The *null space* of $\mathbf{A}$ is the set of all vectors $\mathbf{x}$ where $\mathbf{Ax} = \mathbf{0}$. Sometimes the term *kernel* is used instead of *null space.* It is a subspace of $\mathbb{R}^n$.

(iv) The *left null space* of $\mathbf{A}$ is the set of all vectors $\mathbf{x}$ where $\mathbf{x}^T\mathbf{A} = \mathbf{0}^T$. Because this definition is sometimes confusing, it is often defined equivalently as the set of all vectors $\mathbf{x}$ where $\mathbf{A}^T\mathbf{x} = \mathbf{0}$, which is equivalent to the null space of $\mathbf{A}^T$. The *left null space* of $\mathbf{A}$ is a subspace of $\mathbb{R}^m$.

While these are the subspaces you know currently that are closely associated with a matrix, in the coming chapters we will be introducing some additional spaces that are associated with a matrix. These are the spaces spanned by the so-called *Eigenvalues* and *Eigenvectors* of a matrix. These quantities are introduced in the section below.

1.2 Eigenvalues and Eigenvectors

This week, we will be introducing one of the (perhaps) most useful yet most technical topics in Linear Algebra- *eigenvalues* and *eigenvectors.* Suppose we have a matrix $\mathbf{A}$ where for some particular constant λ and vector $\mathbf{v}$, the following is true:

$$\mathbf{Av} = \lambda\mathbf{v}$$

.

In this case, we can see that transforming the vector $\mathbf{v}$ by the matrix $\mathbf{A}$ is equivalent simply to multiplying the vector $\mathbf{v}$ by a scalar. Put simply, we see that the matrix $\mathbf{A}$ *scales* the vector $\mathbf{v}$ without changing the direction of $\mathbf{v}$. Clearly if $\mathbf{v}$ is $\mathbf{0}$, then any value of λ satisfies this equation. We also observe that in order to satisfy the equation above, $\mathbf{A}$ must be a square matrix. Thus, the real question that we want to ask is:

What values of λ and nonzero values of $\mathbf{v}$ satisfy the equation $\mathbf{Av} = \lambda\mathbf{v}$?

Every value of λ that satisfies the equation above is called an *eigenvalue*, while the set of nonzero values of $\mathbf{v}$ associated with each λ are called *eigenvectors.* In the coming sections, we will begin to illustrate several of the uses and applications of eigenvalues and eigenvectors. This week, however, we will only be covering how to solve for the eigenvectors and eigenvalues of a matrix:

1.3 Solving for the Eigenvalues of a matrix

1. First, we need to find all values λ such that $\mathbf{Ax} = \lambda\mathbf{x}$. This means that:

$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{Ax} - (\lambda\mathbf{I})\mathbf{x} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0.$$

2. To solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ we could do some row reductions with the unknown variable λ, however this would be tedious. Instead, it is better to use our knowledge of determinants to solve:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

3. Solving the determinant equation, we should end up with some polynomial equation of λ set equal to 0. One example of such an equation that would yield eigenvalues 0 and 2 would be:

$$\lambda^3 - 4\lambda^2 + 4\lambda = \quad \Rightarrow \quad (\lambda)(\lambda-2)^2 = 0$$

4. The roots of the polynomial equation above (usually referred to as a *characteristic equation*) are the only values that will satisfy $\mathbf{Ax} = \lambda\mathbf{x}$ and hence are the complete set eigenvalues of $\mathbf{A}$. Note that sometimes 0 is a valid eigenvalue.

1.4 Solving for the Eigenvectors associated with an Eigenvalue

1. For each of the eigenvalues derived from the process above, we substitute the value of λ into the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0$.

2. Observing that we are simply solving for the null space of the matrix $(\mathbf{A} - \lambda\mathbf{I})$, we can use the methods we have learned previously to find basis vectors for the null space of $(\mathbf{A} - \lambda\mathbf{I})$.

3. Note that eigenvectors are not unique unlike eigenvalues, since any scalar multiple of an eigenvector is also an eigenvector. Furthermore, we will also remark that the power of the factored term in the characteristic equation above determines the number of vectors that comprise the basis for the null space of $(\mathbf{A} - \lambda\mathbf{I})$.

2 Frequently Asked Conceptual Questions

1. **I'm solving for the eigenvalues of a matrix, but it looks like the characteristic equation does not have any roots. Is this possible?**

   If you ever get non-real roots to your characteristic equation, it is always a good idea to go back to your determinant calculations and make sure that you derived the proper character equation. Most of the matrices that you will be working with in these sections of the book have only real eigenvalues; however, *it is possible for a real-valued matrix to have imaginary eigenvalues.* In this case, you will need to find the imaginary roots by hand (if the characteristic equation is quadratic) or use a root-finding software package.

   If you want to create matrices that are always guaranteed to have real eigenvalues for extra practice, you could use symmetric matrices (matrices where $\mathbf{A} = \mathbf{A}^T$).

3 Examples

N.B: The examples below are more conceptually oriented, because they tend to be the ones that students have difficulty with. For some calculation-oriented examples, see the textbook.

1. **Let F be the so-called "Fibbonacci" matrix where** $\mathbf{F} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$**.**

(a) The Fibbonacci sequence is a sequence of numbers $1, 1, 2, 3, 5, 8, 13, ...$ where each number is the sum of the previous two numbers, that is, $a_n = a_{n-1} + a_{n-2}$. Show that:

$$\mathbf{F}\begin{bmatrix} a_{n-1} & a_{n-2} \end{bmatrix}^T = \begin{bmatrix} a_n & a_{n-1} \end{bmatrix}^T$$

We simply calculate that:

$$\mathbf{F}\begin{bmatrix} a_{n-1} & a_{n-2} \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} (a_{n-1} + a_{n-2}) \\ (a_{n-1}) \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}.$$

(b) Find all eigenvalues of $\mathbf{F}$.
(*Hint: one of the eigenvalues should be the golden ratio,* $\phi \approx 1.618$).

We first solve:

$$0 = \det(\mathbf{F} - \lambda \mathbf{I_2}) = \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right)$$

Evaluating the determinant and applying the quadratic formula, we get the eigenvalues:

$$0 = -\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1 \qquad \Rightarrow \qquad \lambda = \frac{1 \pm \sqrt{5}}{2}$$

The positive eigenvalue of $\mathbf{F}$, $\lambda = \frac{1+\sqrt{5}}{2}$ is the golden ratio $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$.

Additional References:

I would highly recommend looking into the following resources:

1. *Linear Algebra and Its Applications, 5th Edition* by Lay and McDonald (ISBN-13: 978-0321982384)

2. 3Blue1Brown *Essence of Linear Algebra Series*: www.3blue1brown.com/essence-of-linear-algebra-page

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