MTH 2311 Linear Algebra
Week 8 Resources

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Major Topics:

1. The Four Fundamental Subspaces (Continued)
2. Linearly Independent Sets and Bases

Textbook Material:

Linear Algebra and Its Applications, 5th Edition by Lay and McDonald
Sections 4.2-4.4

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1 Conceptual Review

1.1 The Four Fundamental Subspaces (Continued)

We reviewed last week some of the properties of subspaces and how there are some subspaces closely associated with matrices, such as the *column space*, the *row space*, the null space, and finally the lesser known *left null space* (N.B: for reasons that are clarified below, the textbook does not use the term 'left null space', but rather the 'null space of the transpose'). If we set up some matrix-vector equations, we can provide some more concrete definitions for these four subspaces, which are as follows:

Suppose we have a matrix $\mathbf{A} \in \mathbb{R}^m \times n$ (that is, $\mathbf{A}$ has m rows and n columns). The definitions of the four fundamental subspaces of $\mathbf{A}$ are as follows:

(i) The *column space* of $\mathbf{A}$ is the span of the columns of $\mathbf{A}$, which is the same thing as the range of $\mathbf{A}$. The *row space* of $\mathbf{A}$ is a subspace of $\mathbb{R}^m$.

(ii) The *row space* of $\mathbf{A}$ is the span of the rows of $\mathbf{A}$. Upon closer inspection, this is the same thing as the span of the columns of $\mathbf{A}^T$, which is the same thing as the range of $\mathbf{A}^T$. The *row space* of $\mathbf{A}$ is a subspace of $\mathbb{R}^n$.

(iii) The *null space* of $\mathbf{A}$ is the set of all vectors $\mathbf{x}$ where $\mathbf{Ax} = \mathbf{0}$. Sometimes the term *kernel* is used instead of *null space*. It is a subspace of $\mathbb{R}^n$.

(iv) The *left null space* of $\mathbf{A}$ is the set of all vectors $\mathbf{x}$ where $\mathbf{x}^T\mathbf{A} = \mathbf{0}^T$. Because this definition is sometimes confusing, it is often defined equivalently as the set of all vectors $\mathbf{x}$ where $\mathbf{A}^T\mathbf{x} = \mathbf{0}$, which is equivalent to the null space of $\mathbf{A}^T$. The *left null space* of $\mathbf{A}$ is a subspace of $\mathbb{R}^m$.

While you are not expected to know all of these definitions in these chapters (because there are multiple names and equivalent definitions for each subspace), it is important to at least spend some time considering these spaces and how they relate to one another. (The exact relationship, as interesting as it would be to cover it now, will be coming in future sections).

1.2 Linearly Independent Sets and Bases

By now you may be wondering how we can actually describe some of these spaces that we just defined. In particular, it might be useful to describe mathematically the fundamental subspaces of a given matrix. Because the textbook focuses primarily on the column space (range) and the null space of a matrix, we will be focusing primarily on these two. If you are ever asked to calculate the row space or the left null space of a matrix $\mathbf{A}$, simply rotate your head 90 degrees to the left and re-frame the question as finding the column-space of $\mathbf{A}^T$ or the null space of $\mathbf{A}^T$. (Actually, please don't do that- you can really do a number on your neck).

To describe each of these subspaces we usually provide a *minimal basis* for that subspace. A *minimal basis* is a set of linearly independent vectors such that every vector in in the subspace can be written as a linear combination of those minimal basis vectors. Note that in the special case where our subspace is simply $\{\mathbf{0}\}$, we say that the the *minimal basis* for that subspace is the empty set, denoted by $\emptyset$ or $\{\ \}$. The process for calculating the column space and null space are briefly summarized below. If you want a more in-depth overview of how to calculate minimal bases for these sets, consult your textbook in section 4.3, pg. 202 (null space) or pg. 213 (column space).

How to calculate a minimal basis for the range of a matrix:

1. Row-reduce the matrix into reduced row echelon (or simply row-echelon) form.

2. Identify the pivot columns and group the corresponding columns of the *original* matrix in a set. This is a minimal basis for the column space.

How to calculate a minimal basis for the null space of a matrix:

1. Row-reduce the matrix into reduced row-echelon (or simply row-echelon) form.
2. Equate the rows of the row-reduced matrix to zero (Observe we are solving $\mathbf{Ax} = \mathbf{0}$).
3. For each of these rows, solve for the pivot value of each row, putting all other terms on the other side of the '=' sign.
4. Write the resulting solution to $\mathbf{Ax} = \mathbf{0}$ in vector parametric form, i.e:

$$\begin{bmatrix} x_1 \\ x_2 \\ ... \end{bmatrix} = x_1\mathbf{v_1} + x_2\mathbf{v_2} + ...$$

5. Group all of the vectors $\mathbf{v_1}, \mathbf{v_2}$, etc. into a set. This is a minimal basis for the null space.

For concrete examples of finding a basis for the column space and the null space, see the 'Examples' section below.

2 Frequently Asked Conceptual Questions

1. **Our textbook uses the strange notation of the 'direct sum' operator $\oplus$. What does this mean in terms of subspaces?**

   The $\oplus$ operator denotes that any two subspaces can be combined such that the span of the vectors from both vector spaces produce the resulting vector space. However, it also requires that the intersection of the two 'summand' spaces (the spaces being combined together) have what is called a *trivial intersection*, meaning that the only vector contained in both of the vector spaces is $\mathbf{0}$, which must be a member of both by definition. One example is below:

   Let $U = \text{span}(\{\mathbf{e_1}, \mathbf{e_2}\})$ and $V = \text{span}(\{\mathbf{e_3}\})$ where U, V are subspaces of $\mathbb{R}^3$, and $\mathbf{e_i}$ is the ith natural basis vector (ith column of I_3). Then we can write:

$$U \oplus V \equiv \mathbb{R}^3$$

   To summarize, do no think of the $\oplus$ as an operation (at least in the context of this course), but think of it as part of a statement about subspace structure.

3 Examples

N.B: The examples below are more conceptually oriented, because they tend to be the ones that students have difficulty with. For some calculation-oriented examples, see the textbook.

1. **Let $\mathbf{A}$ be the matrix** $\begin{bmatrix} 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ -4 & -4 & -4 & -8 & -18 \end{bmatrix}$. **Do the following:**

(a) Calculate a minimal basis for the column space (range) of $\mathbf{A}$.

Our first step is to reduce the matrix above into reduced row echelon form:

$$\begin{bmatrix} 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ -4 & -4 & -4 & -8 & -18 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 8/3 & 17/6 \\ 0 & 0 & 1 & -2/3 & 5/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Picking out the columns of the original matrix that are pivot columns in the row-reduced matrix, we get that a minimal basis for the column space is given by the first and third columns of $\mathbf{A}$:

$$\left\{ \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} \right\}$$

(b) Calculate a minimal basis for the null space (kernel) of $\mathbf{A}$.

Equating the row reduced form of the matrix $\mathbf{A}$ to $\mathbf{0}$, we can construct to a system of linear equations that represents the solutions to $\mathbf{Ax} = \mathbf{0}$ (all vectors $\mathbf{x}$ in the null space):

$$\begin{bmatrix} 1 & 1 & 0 & 8/3 & 17/6 \\ 0 & 0 & 1 & -2/3 & 5/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} x_1 + x_2 + (8/3)x_4 + (17/6)x_5 = 0 \\ x_3 + (-2/3)x_4 + x_5 = 0 \end{cases}$$

Solving for the pivot variables in terms of the free variables, we can characterize all solutions to $\mathbf{Ax} = \mathbf{0}$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} (-1)x_2 + (-8/3)x_4 + (-17/6)x_5 \\ x_2 \\ (2/3)x_4 + (-1)x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8/3 \\ 0 \\ 2/3 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -17/6 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

So a minimal basis for the null space is given by:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -8/3 \\ 0 \\ 2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -17/6 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Additional References:

I would highly recommend looking into the following resources:

1. *Linear Algebra and Its Applications, 5th Edition* by Lay and McDonald (ISBN-13: 978-0321982384)

2. 3Blue1Brown *Essence of Linear Algebra Series*: www.3blue1brown.com/essence-of-linear-algebra-page