MTH 2311 Linear Algebra
Week 7 Resources

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Major Topics:

1. Definitions and Applications of Subspaces
2. The Four Fundamental Subspaces

Textbook Material:

Linear Algebra and Its Applications, 5th Edition by Lay and McDonald
Sections 4.1-4.4

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1 Conceptual Review

1.1 Tutor Remark

Due the extended Spring Break, it is important to take some time to review the concepts you have been covering up to this point, in order that you will have a solid understanding going forward. Learning Linear Algebra is like any learning any language- the concepts and structures in this course all rely on each other and require a cumulative knowlege of all underlying content to understand fully. When working on getting back up to speed in this coming week, be sure to take some time to review the basics, because many of the big results in this section of the text (the fundamental subspaces) require a solid understanding of the material you are expected to have mastered by now.

1.2 Subspaces (review)

As a brief review, it is worth recalling the three properties that must hold true for some subset of a vector space to be a vector space itself, which we typically refer to as a 'subspace':

A vector space H that is a subspace of $\mathbb{R}^n$ has the following properties:

(a) There exists a zero vector in H.

(b) For each vector $\mathbf{u}$ and $\mathbf{v}$ in H, the vector $(\mathbf{u}+\mathbf{v})$ is in H.

(c) For each $\mathbf{u}$ in H and real-valued scalar α, $\alpha\mathbf{u}$ is in H.

To summarize, a subspace of a vector space is a subset of a vector space that is closed under addition and closed under scalar multiplication. Some of the misconceptions that students commonly have about vector subspaces are listed below in the FAQ from the week 5 resource, where we introduced subspaces for the first time.

2 Frequently Asked Conceptual Questions

1. **Our textbook uses the strange notation of the 'direct sum' operator $\oplus$. What does this mean in terms of subspaces?**

   The $\oplus$ operator denotes that any two subspaces can be combined such that the span of the vectors from both vector spaces produce the resulting vector space. However, it also requires that the intersection of the two 'summand' spaces (the spaces being combined together) have what is called a *trivial intersection*, meaning that the only vector contained in both of the vector spaces is $\mathbf{0}$, which must be a member of both by definion. One example is below:

   Let $U = \text{span}(\{\mathbf{e_1}, \mathbf{e_2}\})$ and $V = \text{span}(\{\mathbf{e_3}\})$ where U, V are subspaces of $\mathbb{R}^3$, and $\mathbf{e_i}$ is the ith natural basis vector (ith column of I_3). Then we can write:

   $$U \oplus V \equiv \mathbb{R}^3$$

   To summarize, do no think of the $\oplus$ as an operation (at least in the context of this course), but think of it as part of a statement about subspace structure.

2. **I think I have a general idea of whether or not a subspace is a subset of a vector space, but how do I show that the two closure properties hold for all vectors in the subspace?**

   While doing the closure calculations for properties (b) and (c) in the list above is physically impossible to do for all possible numbers at once, it is possible to do these calculations symbolically. That is, use variables! Introducing arbitrary variables is something that some budding mathematicians tend to have a fear of because it creates the illusion of artificially adding to the complexity of the problem. Most of the time, it actually makes proofs easier! I would encourage you to get used to using arbitrary variables, as it is an important skill to exercise when writing proofs or demonstrating some property.

3 Examples

N.B: The examples below are more conceptually oriented, because they tend to be the ones that students have difficuly with. For some calculation-oriented examples, see the textbook.

1. **Let $\mathbf{A}$ be a matrix in $\mathbb{R}^{n \times n}$. Do/Show the following:**

   (a) Every vector in the null space of $\mathbf{A}$ is orthogonal to every vector in the range of $\mathbf{A}^T$ (sometimes called the 'row space' of $\mathbf{A}$).

   (b) Using your answer above, find all vectors in the intersection of the null space of $\mathbf{A}$ and the range of $\mathbf{A}^T$.

   Note: This proof/demonstration will most likely be one of the toughest that you will encounter, so if you have a solid grasp on this, then you should be well-equiped to tackle most other proof-like problems in this course.

   If a vector $\mathbf{y}$ is in the range of $\mathbf{A}^T$, then $\mathbf{y}$ must be some linear combination of the rows of $\mathbf{A}$ (note we are using *rows* not *columns* here, since we are working with $\mathbf{A}^T$, not $\mathbf{A}$). Letting r_i be the ith row of $\mathbf{A}$, we know that if $\mathbf{y}$ is in the range of $\mathbf{A}$, then we can write:

   $$\mathbf{y} = \sum_{i=1}^{n} c_i \mathbf{r_i} \text{ for constants } c_i.$$

   Likewise, if $\mathbf{z}$ is in the null space of $\mathbf{A}$, then we can write $\mathbf{Az} = \mathbf{0}$, so for each $\mathbf{r_i}$, $\mathbf{r_i}^T \mathbf{z} = 0$.

   To show that every vector in range($\mathbf{A}^T$) is orthogonal to every vector in null($\mathbf{A}$), it suffices to show that for every $\mathbf{y}$ and $\mathbf{z}$ as defined above, $\mathbf{y}^T \mathbf{z} = 0$. We can show this with the following:

   $$\mathbf{y}^T \mathbf{z} = \left(\sum_{i=1}^{n} \mathbf{r_i} c_i \right)^T \mathbf{z} = \left(\sum_{i=1}^{n} c_i \mathbf{r_i}^T \right) \mathbf{z} = \sum_{i=1}^{n} c_i \mathbf{r_i}^T \mathbf{z} = \sum_{i=1}^{n} (0) = 0$$

   (c) Find all vectors in null($\mathbf{A}$) $\cap$ range($\mathbf{A}^T$).

   Because null($\mathbf{A}$) $\perp$ (is orthogonal to) range($\mathbf{A}^T$), we observe that the only vector orthogonal to itself is $\mathbf{0}$, and it must be contained in both subspaces. Thus:

   $$\text{null}(\mathbf{A}) \cap \text{range}(\mathbf{A}^T) = \{\mathbf{0}\}.$$

Additional References:

I would highly recommend looking into the following resources:

1. *Linear Algebra and Its Applications, 5th Edition* by Lay and McDonald (ISBN-13: 978-0321982384)

2. 3Blue1Brown *Essence of Linear Algebra Series*: www.3blue1brown.com/essence-of-linear-algebra-page