MTH 2311 Linear Algebra
Week 6 Resources

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3/2/20

Major Topics:

1. Determinants and Determinant Applications

Textbook Material:

Linear Algebra and Its Applications, 5th Edition by Lay and McDonald
Sections 3.1-3.3

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1 Conceptual Review

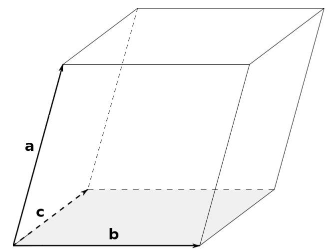
1.1 Determinants

Recall that determinants are used as means of "determining" whether or not a matrix is invertible. Although you might be tempted to limit your understanding of determinants to just this fact, it is important to observe some of the more robust properties of determinants. Before you proceeed through this resource, if you need a refresher on how to calculate determinants, I would highly recommend that you look at the accompanying reference sheet to these resources, which provides a brief overview of how to calculate determinants as well as some other useful quantities in Linear Algebra. While the textbook may be the best reference for reviewing some of these properties, I have provided a list below of some of the properties of determinnts that are either non-trivial or tend to be very useful in solving problems:

1. The determinant of a linear transformation L from $\mathbb{R}^n \to \mathbb{R}^n$ measures the ratio of the n-dimensional volume $L(S)/S$ for some shape S. This is useful in finding the volume

of parallelepipeds (3D parallelograms) whose side vectors represent the columns of a matrix:

$$\det\left(\begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix}\right) = \text{Signed volume of:}$$



If the determinant is 0, it make sense intuitively that the transformation L compresses a n-dimensional space in to a m-dimensional space, where $m < n$. If the determinant is negative, the absolute value still represents volume of the parallelepiped above (see FAQ below for more on this).

2. We can calculate determinants through two main methods, cofactor expansion and diagonal products. If a matrix is in echelon form (e.g: is an upper-triangular or lower-triangular matrix) the determinant of the matrix is simply the product of the main diagonal. If we are given any matrix, we can factor it as the product of a lower and upper triangular matrix (LU factorization) and multiply the determinants of these two matrices. For a refresher on LU factorization, see section 2.5 in the textbook (pp 127-129). An example showing this is below:

$$\begin{aligned} \det(\mathbf{A}) = \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) &= \det(LU) = \det\left(\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\right)\det\left(\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}\right) = (1)(-2) = -2 \end{aligned}$$

Observe that this is significantly faster than the cofactor method!

3. **(Cramer's rule)** Suppose that we have a matrix-vector product equation $\mathbf{Ax} = \mathbf{b}$. We can find the components (say x_i) of the solution vector by calculating the ratio of the determinant of the transformation $\mathbf{A}$ interposed with the solution vector $\mathbf{b}$ and the regular determinant of $\mathbf{A}$. Namely, if $\mathbf{a_i}$ is the ith column of $\mathbf{A}$:

$$x_i = \frac{1}{\det(\mathbf{A})}\det\left(\begin{bmatrix} \mathbf{a_1} & \ldots & \mathbf{a_{i-1}} & \mathbf{b} & \mathbf{a_{i+1}} & \ldots & \mathbf{a_n} \end{bmatrix}\right)$$

While this is a useful result in proving things, it is worth noting that generally this method should not be used to solve linear systems of equations like $\mathbf{Ax} = \mathbf{b}$, since it requires calculating up to $n + 1$ determinants (yikes!). Instead, it is better to row reduce the system.

4. Some additional useful properties of determinants are listed below:

   Given that **A** and **B** are $n \times n$ matrices and α is a scalar:

   (a) $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$

   (b) $\det(\mathbf{A}^{-1}) = \dfrac{1}{\det(\mathbf{A})}$

   (c) $\det(\mathbf{A}^T) = \det(\mathbf{A})$

   (d) $\det(\alpha\mathbf{A}) = \alpha^n \det(\mathbf{A})$

   (e) If $\rho(\bullet)$ is any addition or subtraction of one row from another, then:

   $$\det(\rho(\mathbf{A})) = \det(\mathbf{A})$$

   (f) If $\mu_{i,\alpha}(\bullet)$ multiplies row i of a matrix by α, then:

   $$\det(\mu_{i,\alpha}(\mathbf{A})) = \alpha \det(\mathbf{A})$$

   (g) If $\tau_{i,j}(\bullet)$ denotes a swap of rows i and j of a matrix, then:

   $$\det(\tau_{i,j}(\mathbf{A})) = -\det(\mathbf{A})$$

2 Frequently Asked Conceptual Questions

1. **When should I use the cofactor method over the LU method for calculating a determinant?**

   Although the LU factorization method seems to require more steps for smaller matrices, it is actually faster for large matrices. Most robust software packages like MATLAB use a variant of the LU method for calculating determinants. However there are instances where the cofactor method is faster, especially when there are several zero entries in the matrix (meaning the matrix is *sparse*) or there are identity blocks in a block matrix. Try experimenting with both methods on sparse and matrices with no zero entries- you will see quite quickly why one method is better than the other.

2. **I am attempting to find the volume of a parallelepiped, but I got a negative number when trying to calculate the determinant. What does this mean?**

   Recall that the absolute value of the determinant is what gives the volume of the paralellepiped. However, there is a valid interpretation of both positive and negative determinants, and it has to do with the concept of *signed volume*, in which the order we identify the sides of the parallelepiped matter. Observe that if we were to swap any two column vectors in the matrix we took the determinant of, it would multiply the resulting determinant by -1. One common way of interpreting the sign of the determinant is whether or not the ordering of the columns follows the “righthand rule” (positive determinant) or the unconventional “lefthand rule” (negative determinants).

3 Examples

N.B: The examples below are more conceptually oriented, because they tend to be the ones that students have difficuly with. For some calculation-oriented examples, see the textbook.

1. **Let A be an invertible matrix with determinant $a \neq 0$. Do the following:**

   (a) Show that $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$:

   To show that $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$, it suffices to show that $I = (\mathbf{A}^T)(\mathbf{A}^T)^{-1} = (\mathbf{A}^T)(\mathbf{A}^{-1})^T$. To do this, we show that:

   $$(\mathbf{A}^T)(\mathbf{A}^{-1})^T = (\mathbf{A}\mathbf{A}^{-1})^T = I^T = I.$$

   (b) In terms of a, calculate $\det((\mathbf{A}^T)^{-1})$:

   Using the identity above, we see that:
   $\det((A^T)^{-1}) = \det((A^{-1})^T) = \det(A^{-1}) = 1/\det(\mathbf{A}) = a^{-1}$

   (c) Calculate $\det(\mathbf{A}^{-1}\mathbf{A}^T)$:

   $$\det(\mathbf{A}^{-1}\mathbf{A}^T) = \det(\mathbf{A}^{-1})\det(\mathbf{A}^T) = \frac{1}{\det(\mathbf{A})}\det(\mathbf{A}) = 1$$

   (d) Calculate the value of: $\dfrac{\det(\mathbf{A}^T)}{\det(\mathbf{A}^{-1})}$:

   $$\frac{\det(\mathbf{A}^T)}{\det(\mathbf{A}^{-1})} = \det(\mathbf{A})\det(\mathbf{A}) = a^2$$

2. **Let the matrix $\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$.**

   Calculate the LU factorization of M so that $\mathbf{LU} = \mathbf{M}$, and use this LU factorization to calculate $\det(\mathbf{M})$.

   By reducing M to echelon form, we find that $\mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$.

   So: $\det(\mathbf{M}) = \det(\mathbf{LU}) = \det(\mathbf{L})\det(\mathbf{U}) = (1)(10) = 10$

Additional References:

I would highly recommend looking into the following resources:

1. *Linear Algebra and Its Applications, 5th Edition* by Lay and McDonald (ISBN-13: 978-0321982384)
2. 3Blue1Brown *Essence of Linear Algebra Series*: www.3blue1brown.com/essence-of-linear-algebra-page