MTH 2311 Linear Algebra
Week 5 Resources

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Major Topics:

1. Matrix Applications (Optional material covered in some sections)
2. Vector Spaces and Subspaces

Textbook Material:

Linear Algebra and Its Applications, 5th Edition by Lay and McDonald
Sections 2.2-2.9

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1 Conceptual Review

1.1 Matrix Applications

While some sections of Linear algebra skip over these applications sections, other sections do spend a bit of time working with these applications (because they tend to show up as applications in other courses). I have included the same reference table as last week for your convenience, as well as some more application problems below in the "examples" section. If any of you are engineering, statistics, economics or computer science majors, I would encourage you to skim through some of these sections, because it will expose you to some motivating problems that will show you some of the diverse ways Linear Algebra can be applied to your major:

Form/Factorization	Application	Textbook Reference
Discrete Component Analysis	Circuits (Ohm's Law)	1.10 pg. 83
Networks	Network Flow Analysis	1.10 pg. 85
Block / Partitioned Matrices	(several)	2.4 pgs.119-123
LU Factorization	Circuit Design (among others)	2.5 pgs. 127-129
Leontief I/O Model	Economics	2.6 pgs. 135-137

1.2 Vector Spaces and Subspaces

The concepts of vector spaces, while not inherently difficult to work with and apply, tend to trip people up because they are quite abstract and resorting to concrete examples when explaining them can often result in some misleading assumptions. Thus, I have put together a short review of how we define and work with vector spaces. If you are confused about these concepts, I would highly encourage that you work through the process of *proving* something is a vector space in order to better understand the underlying definitions.

You know by now that for every n, the space $\mathbb{R}^n$ is a vector space upon which we have defined vector additon as well as scalar multiplication. A *subspace* is simply a vector space that is closed under the same operations that define a vector space in the first place. The formal definition that the textbook gives is below:

A subspace H of $\mathbb{R}^n$) has the following properties:

1. There exists a zero vector in H.
2. For each vector $\mathbf{u}$ and $\mathbf{v}$ in H, the vector $(\mathbf{u}+\mathbf{v})$ is in H.
3. For each vector $\mathbf{u}$ in H and real-valued scalar α, $\alpha\mathbf{u}$ is in H.

To word it concisely, a Subspace of a vector space is a *subset* of a vector space that is *closed under addition* and *closed under scalar multiplication*. Some of the misconceptions about vector spaces and subspaces are addressed below in the FAQ section.

2 Frequently Asked Conceptual Questions

1. **I think I have a general idea of whether or not a subspace is a subset of a vector space, but how do I show that the two closure properties hold for all vectors in the subspace?**

   While doing the closure calculations for properties (2) and (3) in the list above is physically impossible to do for all possible numbers at once, it is possible to do these calculations symbolically. That is, use variables! Introducing arbitrary variables is something that some budding mathematicians tend to have a fear of because it creates the illusion that we are artificially adding to the complexity of the problem. Most of the time, it actually makes proofs easier! I would encourage you to get used to using arbitrary variables, as it is an important skill to exercise when writing proofs or demonstrating some property.

2. **Are lower dimensional vector spaces subspaces of higher dimensional vector spaces? In concrete terms, is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?**

   No, lower dimensional vector spaces like $\mathbb{R}^2$ are not technically subspaces of higher dimensional spaces like $\mathbb{R}^3$, simply beccause the number of entries in the vectors are mismatched, and thus one cannot be a subset of the other. However, $\mathbb{R}^2$ could be made a subspace of $\mathbb{R}^3$ by appending zeroes to the bottom of the vectors in $\mathbb{R}^2$:

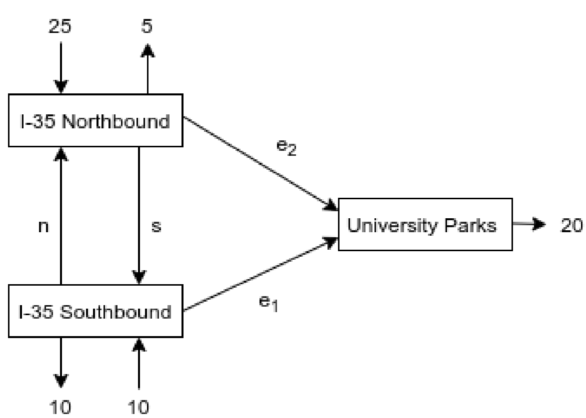
$$\left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \text{ where } x, y \in \mathbb{R} \right\} \text{ is a subspace of } \mathbb{R}^3$$

   This is a great go-to example when explaining the concept of subspaces to students for the first time.

3 Examples

N.B: The examples below are more conceptually oriented, because they tend to be the ones that students have difficuly with. For some calculation-oriented examples, see the textbook.

1. **Suppose that we are asked to predict trafic flow at the I-35 exit ramp onto University Parks Drive during the mornings of Baylor Football Games. Measuring the average number of cars that pass every minute, you come up with the following network flow diagram:**



   (a) Write the following network flow problem as a matrix-vector product equation (i.e. of the form $\mathbf{Ax} = \mathbf{b}$, where the entries of $\mathbf{x}$ are n, s, e_1, and e_2).

Letting the rows denote the I-35 Northbound, I-35 Southbound, and Univ. Parks Nodes respectivey, we get the matrix-vector product:

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} n \\ s \\ e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} (25-5) \\ (10-10) \\ (-20) \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ -20 \end{bmatrix}$$

(b) From this model can we accurately predict the flow of traffic? (i.e. does the equation you wrote down for (a) have a unique solution?)

We observe from the row reduced augmented matrix below that there are many unique solutions to the traffic flow pattern, so we cannot accurately predict the traffic flow:

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 20 \\ 0 & 0 & 1 & 1 & -20 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

2. **Show that the range of a $n \times m$ matrix A is a subspace of $\mathbb{R}^n$**

We observe that $\mathbf{A0} = \mathbf{0}$, so $\mathbf{0}$ is in the range of $\mathbf{A}$, which gives us property (1) from above. Now let $\mathbf{x}$ and $\mathbf{y}$ be in the range of $\mathbf{A}$ so that for vectors $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^m$, $\mathbf{Au} = \mathbf{x}$ and $\mathbf{Av} = \mathbf{y}$. We can show property (2) from above using the linearity of matrix multiplication:

$$\mathbf{x} + \mathbf{y} = \mathbf{Au} + \mathbf{Av} = \mathbf{A}(\mathbf{u} + \mathbf{v}),$$

so clearly $(\mathbf{x} + \mathbf{y})$ is in the range of $\mathbf{A}$. Finally, we show property (3) holds also by the linearity of $\mathbf{A}$. Letting α be a real-valued scalar, we get:

$$\alpha\mathbf{x} = \alpha(\mathbf{Au}) = \mathbf{A}(\alpha\mathbf{u}),$$

So $\alpha\mathbf{x}$ is in the range of $\mathbf{A}$, which shows that the range of $\mathbf{A}$ is a subspace of $\mathbb{R}^n$

Additional References:

I would highly recommend looking into the following resources:

1. *Linear Algebra and Its Applications, 5th Edition* by Lay and McDonald (ISBN-13: 978-0321982384)

2. 3Blue1Brown *Essence of Linear Algebra Series*: www.3blue1brown.com/essence-of-linear-algebra-page

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