MTH 2311 Linear Algebra
Week 4 Resources

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Major Topics:

1. Matrix Operations and Properties (Continued)
2. Matrix Forms and Factorizations

Textbook Material:

Linear Algebra and Its Applications, 5th Edition by Lay and McDonald
Sections 1.10-2.5

Author's Note

This far in the semester, you should feel confident in the more mechanical aspects of Linear Algebra, as you have already mastered basic matrix operations and some of the foundational theory. Also, at this point most sections (including your own, hopefully) will have taken their first exam. From this point forward, the material becomes more concept-oriented and less calculation-oriented. You will be seeing less problems that begin with "Calculate ..." and more problems that begin with "Prove ..." or "Show that ...". Thus, it is important that you are willing to invest the time in building a deep understanding of the concepts you will be encountering in the second section of the textbook and beyond. Linear Algebra is very much a cumulative subject, and in the coming chapters, you will begin to realize how each of the isolated concepts you have been learning so far are all connected through several key theorems. Be sure to stay engaged and remind yourself that with a bit of perseverance you can and will do well in this course!

–Colin B.

1 Conceptual Review

1.1 Matrix Operations and Properties (Continued)

We discussed last week the concept of invertibility. To put the concept of invertibility in the context of what we have learned so far, we can recognize a matrix as invertible by seeing that it satisfies at least one element in the TFAE (*The Following Are Equivalent*) list below:

The Invertible Matrix theorem: For any square $n \times n$ matrix $\mathbf{A}$ the following are equivalent:

1. $\mathbf{A}$ is invertible.
2. $\mathbf{A}$ has a nonzero determinant.
3. $\mathbf{A}$ is a full rank matrix.
4. $\mathbf{A}$ is row-equivalent to the identity matrix I
5. The equation $\mathbf{Ax} = \mathbf{0}$ only has the trivial solution $\mathbf{x} = \mathbf{0}$
6. The columns of $\mathbf{A}$ are linearly independent.
7. The rows of $\mathbf{A}$ are linearly independent.
8. The range of $\mathbf{A}$ (when multiplied by a vector on the right) is $\mathbb{R}^n$, making $\mathbf{A}$ 1-1 and onto.
9. The range of $\mathbf{A}^T$ (when multiplied by a vector on the right) is $\mathbb{R}^n$, making $\mathbf{A}^T$ 1-1 and onto.
10. There is a unique solution to $\mathbf{Ax} = \mathbf{b}$ for any $\mathbf{b} \in \mathbb{R}^n$
11. There is a unique solution to $\mathbf{A}^T\mathbf{x} = \mathbf{b}$ for any $\mathbf{b} \in \mathbb{R}^n$

This may look like lot to digest, but by now you might already have the intuition to see *why* most of the items above are true. In the textbook, this theorem does not appear in its full gory detail until the latter part of section 3 (with even more additions to the list!), but there are several bits and pieces of it scattered throughout section 2, and it helps to begin to observe the overlap of these concepts sooner rather than later.

While the invertible matrix theorem has a reputation among students as “The great laundry-list of Linear Algebra”, it is quite useful, especially now that you are moving away from calculation-oriented problems and more into proof-oriented problems. Because the invertible matrix theorem equates so many of the concepts that you should have a solid grasp on, it is quite useful for proving things. Thus, it might be a good idea to be sure that you take the time to invest in a solid understanding of it before moving on.

1.2 Matrix Forms and Factorizations

There are several additional concepts that tend to come up in application problems; in particular these concepts deal with different forms and factorizations of a matrix. Depending on the type of problem, different forms or factorizations may be useful. Although most sections of Linear Algebra do not cover all of these applications in detail, I have created a short table of references in the textbook, in case you need a referesher:

Form/Factorization	**Application**	**Textbook Reference**
Discrete Component Analysis	Circuits (Ohm's Law)	1.10 pg. 83
Networks	Network Flow Analysis	1.10 pg. 85
Block / Partitioned Matrices	(several)	2.4 pgs.119-123
LU Factorization	Circuit Design (among others)	2.5 pgs. 127-129
Leontief I/O Model	Economics	2.6 pgs. 135-137

2 Frequently Asked Conceptual Questions

1. **In the Invertible Matrix Theorem there seems to be a connection not only between a matrix and its inverse, but also the matrix transpose and its inverse. How are these four matrices related?**

   Interestingly, there is a connection between square invertible matrices and their transposes. While the depth of the connection is not fully explored in this course, we will later touch on the concept of rank and nullity of a matrix, in which we see a connection between the range of a matrix and the null space of the matrix transpose. In fact, these vector subspaces are actually orthogonal to eachother.

2. **I'm stuck trying to show some property about matrices. How should I go about finding a solution to these proof-like problems?**

   In problems like these, a great place to start is to make a list of three things, first *what we are given to be true*, second, *what we are trying to show*, and third, *what we think will be useful in getting from the premise to the conclusion.* For the problems in section 2, it is likely that it will involve the invertible matrix theorem in some form. Metaphorically, you can think of the invertible matrix theorem as a sort of "proof highway" that equates several different concepts. Also, try working backwards from the conclusion as well as forwards from the premise; it's a common technique that mathematicians use when trying to prove something new.

3 Examples

N.B: The examples below are more conceptually oriented, because they tend to be the ones that students have difficuly with. For some calculation-oriented examples, see the textbook.

1. **Show that for an invertible matrix A,** $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$

   Since $\mathbf{A}$ is invertible, it must be square so it seems feasible that $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$. To prove it, we use the fact that the inverse of a square matrix $\mathbf{M}$ satisfies $\mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$:
   This shows that the inverse of $(\mathbf{A}^T)$, $(\mathbf{A}^T)^{-1}$ is equivalent to $(\mathbf{A}^{-1})^T$.

$$\mathbf{A}^T(\mathbf{A}^{-1})^T = (\mathbf{A}^{-1}\mathbf{A})^T = (\mathbf{I})^T = \mathbf{I}$$

2. **Suppose** $L(\mathbf{x}): \mathbb{R}^n \to \mathbb{R}^n$ **is a linear transformation that is onto. How many solutions are there to the equation** $L(L(\mathbf{x})) = \mathbf{b}$**?**

   This problem is an exercise in understanding the invertible matrix theorem. Because L is a linear transformation we know there must be a matrix $\mathbf{L}$ where $L(\mathbf{x}) = \mathbf{Lx}$. We also know that the transformation L is onto. Because the domain and codomain of L are both $\mathbb{R}^n$ then L must also be 1-1. Because L is 1-1 and onto, then the matrix $\mathbf{L}$ is invertible, which means there exists a $n \times n$ matrix $\mathbf{L}^{-1}$ so that:

$$L(L(\mathbf{X})) = \mathbf{LLx} = \mathbf{b} \quad \Rightarrow \quad \mathbf{x} = \mathbf{L}^{-1}\mathbf{L}^{-1}\mathbf{b},$$

   which is the unique solution to $L(L(\mathbf{x})) = \mathbf{b}$

Additional References:

I would highly recommend looking into the following resources:

1. *Linear Algebra and Its Applications, 5th Edition* by Lay and McDonald (ISBN-13: 978-0321982384)
2. 3Blue1Brown *Essence of Linear Algebra Series*: www.3blue1brown.com/essence-of-linear-algebra-page

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