MTH 1320 – PRECALCULUS
FALL 2020 WEEK 6 RESOURCE

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Welcome back, Precalculus scholars! This resource focuses on material covered in the sixth week of classes, namely graphs of polynomials and dividing polynomials from Sections 3.4-3.5 of OpenStax's *Precalculus*. Please refer to previous resources if you would like to review other topics. Don't forget to sign up for Group Tutoring on Tuesdays at 5:00!

3.4 Graphs of Polynomial Functions

Graphs of polynomial functions are **smooth**, with no points or corners (cusps), and **continuous**, with no breaks. For example, the absolute value function, $f(x) = |x|$, is continuous, but it is not a polynomial because its graph comes to a sharp point at $x = 0$. It is also important to remember that a polynomial's domain is all real numbers.

The Graphing Process

To graph a polynomial function, you can follow the steps from OpenStax's *Precalculus* in Figure 1.

HOW TO

Given a polynomial function, sketch the graph.

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetrical about the y-axis, that is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.
4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

Figure 1. Steps to Graph a Polynomial
Source: [1]

Let's dive a little deeper into a couple of these steps.

Factoring Techniques

To find the *x*-intercepts or zeros of a polynomial, you will often need to factor it. Factoring higher order polynomials is similar to factoring some quadratic functions. Some important factoring methods are:

- Pulling out the greatest common factor (largest number or power of x that is in all the terms)
- Factoring binomials that are differences of squares (e.g. $x^2 - 9 = (x+3)(x-3)$)
- Factoring trinomials, polynomials with three terms (if you are having trouble, take a look at this page: https://www.montereyinstitute.org/courses/DevelopmentalMath/COURSE_TEXT2_RESOURCE/U12_L2_T1_text_final.html#:~:text=Trinomials%20in%20the%20form%20x,Example)
- Grouping the terms into groups of two, pulling out the greatest common factor of each group, and pulling out the common factor between the groups (for four-term polynomials)

Once you have factored the polynomial, set each term equal to zero and solve for x to find its zeros.

Zeros' Multiplicities
The **multiplicity** of a zero/factor is the "number of times a given factor appears in the factored form of the equation of a polynomial" [1]. A zero's multiplicity affects the behavior of the graph around that zero. For a zero with a multiplicity of one, the graph will cross the *x*-axis in an approximately straight line. An even multiplicity results in the graph touching but not crossing over the *x*-axis at that zero. An odd multiplicity results in the graph crossing through the *x*-axis with a shape similar to the cubic toolkit function. As a zero's multiplicity increases, the graph becomes flatter around the zero. See Figure 2 for examples of the effects of various multiplicities.

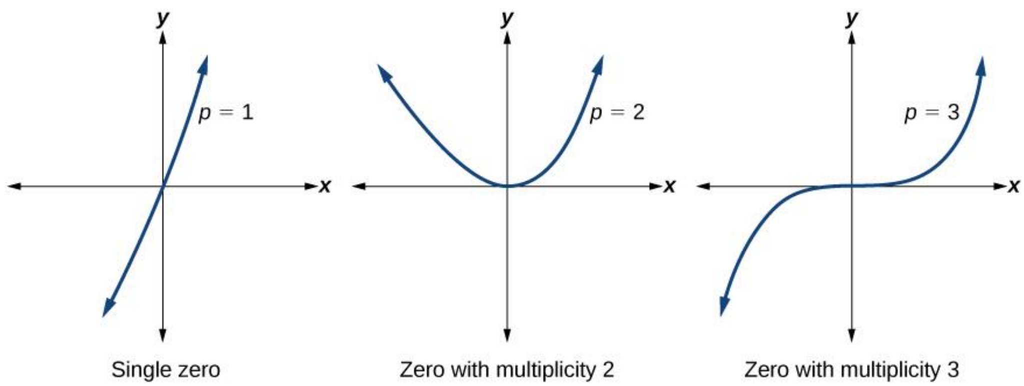

Figure 2. Effects of Different Multiplicities on the Graph of a Polynomial
Source: [1]

The Intermediate Value Theorem
An important theorem that applies to polynomials is the **Intermediate Value Theorem**. It says, "for two numbers a and b in the domain of f, if $a < b$ and $f(a) \neq f(b)$, then the function f takes on every value between $f(a)$ and $f(b)$" [1]. This theorem implies that if $f(a)$ is positive and $f(b)$ is negative, or vice versa, there is at least one value of x between them that is a zero of the polynomial.

3.5 Dividing Polynomials
Sometimes it is necessary or beneficial to divide two polynomials. There are two methods for dividing polynomials: long division and synthetic division. **Long division** of polynomials is very similar to the long division you have learned for decimal numbers. **Synthetic division** is a shortcut method for long division. While synthetic division is quicker than long division, it is also important to learn how to do long division because synthetic division cannot be used for all polynomial division. Synthetic division can only be used when the divisor (the denominator or part that divides) has the form $x - k$, where k is a constant. Let's look at the process for each of these division methods.

Long Division
We can follow these steps to divide a polynomial by another polynomial of lesser degree:

1. Place the numerator or dividend under the long division symbol $\overline{)}$. If any terms of the polynomial are "missing" (i.e. the polynomial skips a power of x), write those terms in with zero as the coefficient.
2. Place the denominator or divisor to the left of the long division symbol $\overline{)}$.
3. Divide the first term of the dividend by the first term of the divisor, writing their quotient above the long division symbol $\overline{)}$.

4. Multiply the quotient term you just wrote by the whole divisor, writing their product below the like terms of the dividend.
5. Subtract the product you just wrote from the terms above it. Don't forget to distribute the minus sign to all the terms!
6. Pull down the next term of the dividend so that it is next to the difference you just got. These terms act as the "dividend" for the next step.
7. Repeat steps 3-7 for the rest of the terms of the dividend.
8. Divide the remainder by the divisor and add it to the rest of the quotient.

Synthetic Division

To perform synthetic division, we only use the coefficients of the dividend as our "dividend", and our "divisor" is the zero k of the divisor $x - k$. This means that you need to switch the sign of the "divisor". Figure 3 from OpenStax's *Precalculus* shows the steps for synthetic division. Note that you must again include zeros for any missing terms of the dividend.

HOW TO

Given two polynomials, use synthetic division to divide.

1. Write k for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by k. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by k. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 0, the next number from the right has degree 1, and so on.

Figure 3. Steps for Synthetic Division
Source: [1]

Example Problems

1. Sketch the graph of the polynomial $f(x) = x^6 - 3x^4 + 2x^2$.

1. "Find the intercepts."

a. Determine the greatest common factor.
   All the terms have x^2 as a factor.
b. Factor out the greatest common factor.
   $f(x) = x^2(x^4 - 3x^2 + 2)$
c. Factor the trinomial.
   We want two numbers with a product of 2 and a sum of -3, which are -1 and -2.
   $f(x) = x^2(x^2 - 1)(x^2 - 2)$
d. Factor the differences of squares. Note that $2 = \left(\sqrt{2}\right)^2$.
   $f(x) = x^2(x - 1)(x + 1)\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$
e. Set the equation equal to zero and solve for x to find the ***x*-intercepts**.
   $f(x) = x^2(x - 1)(x + 1)\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right) = 0$
   $x^2 = 0$ or $x - 1 = 0$ or $x + 1 = 0$ or $x - \sqrt{2} = 0$ or $x + \sqrt{2} = 0$
   $\boldsymbol{x = 0}$ **or** $\boldsymbol{x = 1}$ **or** $\boldsymbol{x = -1}$ **or** $\boldsymbol{x = \sqrt{2}}$ **or** $\boldsymbol{x = -\sqrt{2}}$

f. Evaluate the function at $x = 0$ to find the **y-intercept**. (This can be done using the original or factored form of the function.)
$\boldsymbol{f(0) = 0^2(0^2 - 1)(0^2 - 2) = 0}$

2. **"Check for symmetry."**
The general form of the function only has even powers of x, so it is an **even function**. We can confirm this by checking that $f(-x) = f(x)$.
Evaluating $f(x)$ at $-x$ gives us
$$f(-x) = (-x)^6 - 3(-x)^4 + 2(-x)^2.$$
When we have a product raised to an exponent, we can apply the exponent to each factor of the product, so let's separate each coefficient of -1 from its x.
$$f(-x) = (-1)^6x^6 - 3(-1)^4x^4 + 2(-1)^2x^2$$
We know that any even power of -1 is 1, so we can simplify this to
$$f(-x) = 1 * x^6 - 3 * 1 * x^4 + 2 * 1 * x^2 = x^6 - 3x^4 + 2x^2$$
Since this is our original function, $f(-x) = f(x)$, and the function is in fact even.

3. **"Use the multiplicities of the zeros to determine the behavior of the polynomial at the *x*-intercepts."**
From step 1 part d, the factored form of the polynomial is
$$f(x) = x^2(x - 1)(x + 1)(x - \sqrt{2})(x + \sqrt{2}).$$
Therefore, we have the following multiplicities and corresponding local behavior:
   - **The factor $x = 0$ has a multiplicity of 2**, so the graph will bounce off the *x*-axis there.
   - **All the other factors have a multiplicity of 1**, so the graph will cross through the *x*-axis in approximately a straight line at $x = -\sqrt{2}, -1, 1, \sqrt{2}$.

4. **"Determine the end behavior by examining the leading term."**
The leading term of the polynomial is x^6. An even powered leading term with a positive coefficient (in this case 1), will cause the output to approach infinity as the input approaches infinity or negative infinity. In arrow notation,
$$\textbf{as } \boldsymbol{x \to -\infty, f(x) \to \infty}$$
$$\textbf{as } \boldsymbol{x \to \infty, f(x) \to \infty}$$

5. **"Use the end behavior and the behavior at the intercepts to sketch a graph."**

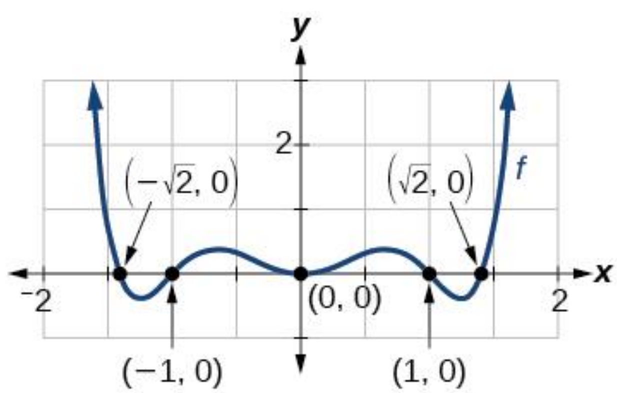

Figure 4. Graph of $f(x) = x^6 - 3x^4 + 2x^2$
Source: [1]

6. **"Ensure that the number of turning points does not exceed one less than the degree of the polynomial."**

The degree of the polynomial is 6, so the number of turning points cannot exceed $6 - 1 = 5$. Our graph has 2 local maxima and 3 local minima, giving us a total of **5 turning points**. Therefore, we do not have too many turning points.

2. Divide $x^3 - 3x + 2$ by $x + 2$ using both long division and synthetic division.

Long division:

$$x+2\overline{)x^3+0x^2-3x+2}$$

$$\begin{array}{l} \quad\quad\;\; x^2 \\ x+2\overline{)x^3+0x^2-3x+2} \end{array}$$

$$\begin{array}{l} \quad\quad\;\; x^2 \\ x+2\overline{)x^3+0x^2-3x+2} \\ \quad\quad\;\; x^3+2x^2 \end{array}$$

$$\begin{array}{l} \quad\quad\;\; x^2 \\ x+2\overline{)x^3+0x^2-3x+2} \\ \quad\;\; \underline{-(x^3+2x^2)} \\ \quad\quad\quad\;\; -2x^2 \end{array}$$

$$\begin{array}{l} \quad\quad\;\; x^2 \\ x+2\overline{)x^3+0x^2-3x+2} \\ \quad\;\; \underline{-(x^3+2x^2)} \;\downarrow \\ \quad\quad\quad\;\; -2x^2-3x \end{array}$$

$$\begin{array}{l} \quad\quad\;\; x^2-2x \\ x+2\overline{)x^3+0x^2-3x+2} \\ \quad\;\; \underline{-(x^3+2x^2)} \\ \quad\quad\quad\;\; -2x^2-3x \end{array}$$

$$\begin{array}{l} \quad\quad\;\; x^2-2x \\ x+2\overline{)x^3+0x^2-3x+2} \\ \quad\;\; \underline{-(x^3+2x^2)} \\ \quad\quad\quad\;\; -2x^2-3x \\ \quad\quad\quad\;\; -2x^2-4x \end{array}$$

$$\begin{array}{l} \quad\quad\;\; x^2-2x \\ x+2\overline{)x^3+0x^2-3x+2} \\ \quad\;\; \underline{-(x^3+2x^2)} \\ \quad\quad\quad\;\; -2x^2-3x \\ \quad\quad\; \underline{-(-2x^2-4x)} \\ \quad\quad\quad\quad\quad\quad\quad\; x \end{array}$$

$$
\begin{array}{r}
x^2 - 2x \phantom{{}+ 1} \\
x + 2 \overline{) x^3 + 0x^2 - 3x + 2} \\
-(x^3 + 2x^2) \phantom{{}- 3x + 2} \\
\hline
-2x^2 - 3x \phantom{{}+ 2} \\
-(-2x^2 - 4x) \phantom{{}+ 2} \\
\hline
x + 2
\end{array}
$$

$$
\begin{array}{r}
x^2 - 2x + 1 \\
x + 2 \overline{) x^3 + 0x^2 - 3x + 2} \\
-(x^3 + 2x^2) \phantom{{}- 3x + 2} \\
\hline
-2x^2 - 3x \phantom{{}+ 2} \\
-(-2x^2 - 4x) \phantom{{}+ 2} \\
\hline
x + 2
\end{array}
$$

$$
\begin{array}{r}
x^2 - 2x + 1 \\
x + 2 \overline{) x^3 + 0x^2 - 3x + 2} \\
-(x^3 + 2x^2) \phantom{{}- 3x + 2} \\
\hline
-2x^2 - 3x \phantom{{}+ 2} \\
-(-2x^2 - 4x) \phantom{{}+ 2} \\
\hline
x + 2 \\
x + 2
\end{array}
$$

$$
\begin{array}{r}
x^2 - 2x + 1 \\
x + 2 \overline{) x^3 + 0x^2 - 3x + 2} \\
-(x^3 + 2x^2) \phantom{{}- 3x + 2} \\
\hline
-2x^2 - 3x \phantom{{}+ 2} \\
-(-2x^2 - 4x) \phantom{{}+ 2} \\
\hline
x + 2 \\
-(x + 2) \\
\hline
0
\end{array}
$$

In this case, we have a remainder of zero, so our quotient is simply

$$x^2 - 2x + 1.$$

Synthetic division:

$$
\begin{array}{r|rrrr}
-2 & 1 & 0 & -3 & 2 \\
 & & & & \\
\hline
\end{array}
$$

$$
\begin{array}{r|rrrr}
-2 & 1 & 0 & -3 & 2 \\
 & \downarrow & & & \\
\hline
 & 1 & & &
\end{array}
$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -3 & 2 \\ & & -2 & & \\ \hline & 1 & & & \end{array}$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -3 & 2 \\ & & -2 & & \\ \hline & 1 & -2 & & \end{array}$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -3 & 2 \\ & & -2 & 4 & \\ \hline & 1 & -2 & & \end{array}$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -3 & 2 \\ & & -2 & 4 & \\ \hline & 1 & -2 & 1 & \end{array}$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -3 & 2 \\ & & -2 & 4 & -2 \\ \hline & 1 & -2 & 1 & \end{array}$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -3 & 2 \\ & & -2 & 4 & -2 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

Going from right to left, we have the remainder, constant, coefficient of x, and coefficient of x^2. Therefore, our quotient is

$$x^2 - 2x + 1$$

Reminders

Thank you for reading this resource! Here are a few reminders about other Precalculus opportunities:

- Group Tutoring for Precalculus is on Tuesdays from 5:00 to 6:00 p.m. through Microsoft Teams. If you have not received an email inviting you to sign up, and you want to attend, please email me at Sydney_Schirner1@baylor.edu, and I will make sure you are included!
- The Tutoring Center has great Precalculus videos on YouTube! You can find links for them at https://www.baylor.edu/support_programs/index.php?id=955694.
- If you want individual help, you can also schedule a FREE 30-minute appointment with one of our awesome Precalculus tutors at https://www.baylor.edu/tutoring.

References

[1] J. Abramson *et al.*, *Precalculus*. Houston, Texas, USA: Rice Univ., 2014. [Online]. Available: https://openstax.org/details/books/precalculus