QUALIFYING EXAM IN REAL VARIABLES
BAYLOR UNIVERSITY
SPRING 2007

Complete four of the following six questions.

1. State precisely the following:

a) Fatou's Lemma

b) The Lebesgue Monotone Convergence Theorem

c) The Hahn-Banach Theorem

d) The Fubini Theorem

e) The Closed Graph Theorem

f) The Krein-Milman Theorem

g) The Riesz Representation Theorem

h) The Radon-Nikodym Theorem

2. Let $f \in L^1[0,1]$. Must

$$\lim_{\alpha \to \infty} \alpha \cdot |\{x : |f(x)| > \alpha\}| = 0 \quad ?$$

Justify your answer.

3. What is the Cantor-Lebesgue function? How may it be used to construct a set S which is Lebesgue measurable but not a Borel set?

4. Let $f_n(t) = \sin nt \quad (-\pi \leq t \leq \pi)$. Show that if $1 \leq p < \infty$ then $f_n \to 0$ weakly in $L^p(-\pi, \pi)$ but not strongly in $L^p(-\pi, \pi)$.

5. a) Is $\mathbb{Q}$ a G_δ ? Justify your answer.

   b) Is there a real-valued function on $\mathbb{R}$ which is continuous on $\mathbb{Q}$ but discontinuous on $\mathbb{R} - \mathbb{Q}$? Justify your answer.

6. a) Let a, b be nonnegative numbers, and p, q such that $1 < p < \infty$ and $1/p + 1/q = 1$. Establish Young's inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

   b) Using Young's Inequality prove the Hölder inequality: If $f \in L^p[0,1]$ and $g \in L^q[0,1]$,where p and q are as above, then $fg \in L^1[0,1]$ and

$$\int |fg| \leq \|f\|_p \cdot \|g\|_q.$$

   c) For $1 < p < \infty$ and $g \in L^q$, consider the linear functional F on L^p given by

$$F(f) = \int fg.$$

   Show that $\|F\| = \|g\|_q$.