1) Let $M$ and $N$ be normal subgroups of $G$ such that $G = MN$. Prove that $G/(M \cap N) \cong (G/M) \times (G/N)$.

2) Let $G$ be a group with $|G| = pqr$ where $p, q, r$ are primes with $p < q < r$. Prove that $G$ has a nontrivial normal subgroup.

3) Solve the simultaneous system of congruences

\[ x \equiv 1 \mod 8 \quad x \equiv 2 \mod 25 \quad x \equiv 3 \mod 81 . \]

4) Show that $p(x) = x^3 + 9x + 6$ is irreducible in $\mathbb{Q}[x]$. Let $\theta$ be a root of $p(x)$. Find the inverse of $1 + \theta$ in $\mathbb{Q}(\theta)$.

5) Let $K$ be a field and $V$ a vector space over $K$ of dimension $n$. Let $A \in \text{End}_K(V)$. Show the following are equivalent:
   a) the minimal polynomial of $A$ is the same as the characteristic polynomial of $A$.
   b) there exists a vector $v \in V$ such that $v, Av, \ldots, A^{n-1}v$ is a basis of $V$.

6) Let $K$ be the splitting field of the polynomial $x^4 - 2$ over $\mathbb{Q}$ and let $G$ be the Galois group of $K$ over $\mathbb{Q}$.
   a) Describe $G$.
   b) Describe all subfields of $K$ containing $\mathbb{Q}$.